COURSE MATERIAL

IV Year B. Tech I- Semester

MECHANICAL ENGINEERING

AY: 2025-26

FINITE ELEMENT METHODS

R22A0326

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MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

DEPARTMENT OF MECHANICAL ENGINEERING

(Autonomous Institution-UGC, Govt. of India)

Secunderabad-500100,Telangana State, India.

www.mrcet.ac.in

CONTENTS

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

VISION

- To establish a pedestal for the integral innovation, team spirit, originality and competence in the students, expose them to face the global challenges and become technology leaders of Indian vision of modern society.

MISSION

- To become a model institution in the fields of Engineering, Technology and Management.
- To impart holistic education to the students to render them as industry ready engineers.
- To ensure synchronization of MRCET ideologies with challenging demands of International Pioneering Organizations.

QUALITY POLICY

- To implement best practices in Teaching and Learning process for both UG and PG courses meticulously.
- To provide state of art infrastructure and expertise to impart quality education.
- To groom the students to become intellectually creative and professionally competitive.
- To channelize the activities and tune them in heights of commitment and sincerity, the requisites to claim the never - ending ladder of **SUCCESS** year after year.

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VISION

To become an innovative knowledge center in mechanical engineering through state-of-the-art teaching-learning and research practices, promoting creative thinking professionals.

MISSION

The Department of Mechanical Engineering is dedicated for transforming the students into highly competent Mechanical engineers to meet the needs of the industry, in a changing and challenging technical environment, by strongly focusing in the fundamentals of engineering sciences for achieving excellent results in their professional pursuits.

Quality Policy

- ✓ To pursuit global Standards of excellence in all our endeavors namely teaching, research and continuing education and to remain accountable in our core and support functions, through processes of self-evaluation and continuous improvement.
- ✓ To create a midst of excellence for imparting state of art education, industry-oriented training research in the field of technical education.

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Department of Mechanical Engineering

PROGRAM OUTCOMES

Engineering Graduates will be able to:

1. **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and teamwork**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

12. Life-long learning: Recognize the need for and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSOs)

PSO1 Ability to analyze, design and develop Mechanical systems to solve the Engineering problems by integrating thermal, design and manufacturing Domains.

PSO2 Ability to succeed in competitive examinations or to pursue higher studies or research.

PSO3 Ability to apply the learned Mechanical Engineering knowledge for the Development of society and self.

Program Educational Objectives (PEOs)

The Program Educational Objectives of the program offered by the department are broadly listed below:

PEO1: PREPARATION

To provide sound foundation in mathematical, scientific and engineering fundamentals necessary to analyze, formulate and solve engineering problems.

PEO2: CORE COMPETANCE

To provide thorough knowledge in Mechanical Engineering subjects including theoretical knowledge and practical training for preparing physical models pertaining to Thermodynamics, Hydraulics, Heat and Mass Transfer, Dynamics of Machinery, Jet Propulsion, Automobile Engineering, Element Analysis, Production Technology, Mechatronics etc.

PEO3: INVENTION, INNOVATION AND CREATIVITY

To make the students to design, experiment, analyze, interpret in the core field with the help of other inter disciplinary concepts wherever applicable.

PEO4: CAREER DEVELOPMENT

To inculcate the habit of lifelong learning for career development through successful completion of advanced degrees, professional development courses, industrial training etc.

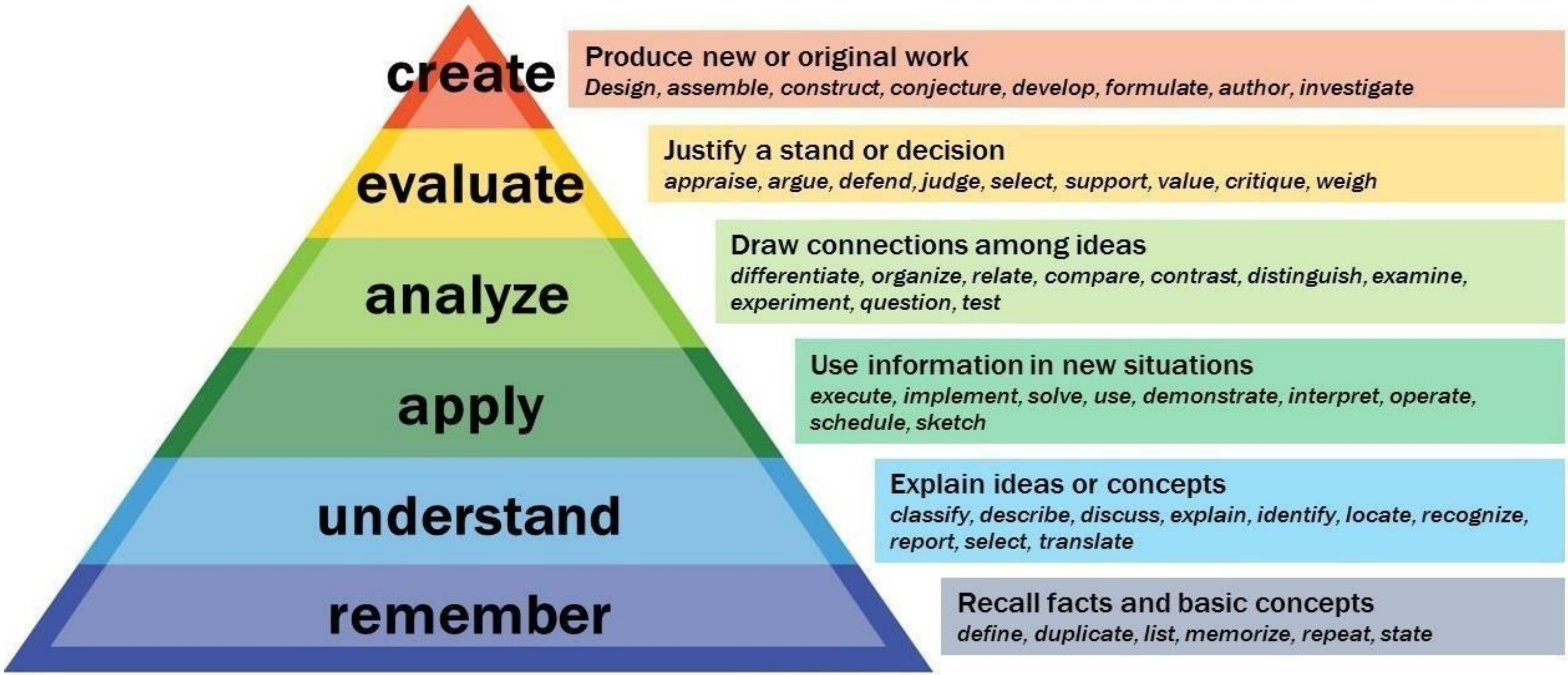
PEO5: PROFESSIONALISM

To impart technical knowledge, ethical values for professional development of the student to solve complex problems and to work in multi-disciplinary ambience, whose solutions lead to significant societal benefits.

Blooms Taxonomy

Bloom's Taxonomy is a classification of the different objectives and skills that educators set for their students (learning objectives). The terminology has been updated to include the following six levels of learning. These 6 levels can be used to structure the learning objectives, lessons, and assessments of a course.

1. **Remembering**: Retrieving, recognizing, and recalling relevant knowledge from long- term memory.
2. **Understanding**: Constructing meaning from oral, written, and graphic messages through interpreting, exemplifying, classifying, summarizing, inferring, comparing, and explaining.
3. **Applying**: Carrying out or using a procedure for executing or implementing.
4. **Analyzing**: Breaking material into constituent parts, determining how the parts relate to one another and to an overall structure or purpose through differentiating, organizing, and attributing.
5. **Evaluating**: Making judgments based on criteria and standard through checking and critiquing.
6. **Creating**: Putting elements together to form a coherent or functional whole; reorganizing elements into a new pattern or structure through generating, planning, or producing.



create
Produce new or original work
Design, assemble, construct, conjecture, develop, formulate, author, investigate
evaluate
Justify a stand or decision
appraise, argue, defend, judge, select, support, value, critique, weigh
analyze
Draw connections among ideas
differentiate, organize, relate, compare, contrast, distinguish, examine, experiment, question, test
apply
Use information in new situations
execute, implement, solve, use, demonstrate, interpret, operate, schedule, sketch
understand
Explain ideas or concepts
classify, describe, discuss, explain, identify, locate, recognize, report, select, translate
remember
Recall facts and basic concepts
define, duplicate, list, memorize, repeat, state

Course Syllabus

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

IV Year B. TECH - I- SEM **L/T/P/C**

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(R22A0326) Finite Element Methods

COURSE OBJECTIVES

The general objectives of the course are to enable the students to

1.Introduce basic concepts of finite element methods including domain discretization, polynomial interpolation and application of boundary conditions

2. Understand the theoretical basics of governing equations and convergence criteria of finite element method.
3. Develop of mathematical model for physical problems and concept of discretization of continuum.
4. To learn the application of FEM equations for Iso-Parametric and heat transfer problems and Discuss the accurate Finite Element Solutions for the various field problems
5. Use the commercial Finite Element packages to build Finite Element models and solve a selected range of engineering problems.

UNIT-I

FUNDAMENTAL CONCEPTS & ONE-DIMENSIONAL PROBLEM: Introduction to Finite Element Method for solving field problems, Stress and Equilibrium, Strain – Displacement relations, Stress- Strain relations. One -Dimensional Problem: Finite element modeling, local coordinates and shape functions. Potential Energy approach, Assembly of Global stiffness matrix and load vector. Finite element equations, Treatment of boundary conditions.

UNIT-II

Trusses: Element matrices, assembling of global stiffness matrix, solution for displacements, reaction, stresses. BEAMS: Element matrices, assembling of global stiffness matrix, solution for displacements, reaction, stresses.

UNIT-III

Two Dimensional Problems: Basic concepts of plane stress and plane strain, stiffness matrix of CST element, finite element solution of plane stress problems. Axi-Symmetric Model: Finite element modeling of axi-symmetric solids subjected to axi- symmetric loading with triangular elements.

UNIT-IV

Iso-Parametric Formulation: Concepts, sub parametric, super parametric elements, two dimensional four nodes iso-parametric elements, and numerical integration. Heat Transfer Problems: One dimensional steady state analysis composite wall. One dimensional fin analysis and two dimensional of thin plate.

UNIT-V

DYNAMIC ANALYSIS: Formulation of finite element model, element matrices, evaluation of Eigen values and Eigen vectors for a stepped bar and a beam.

TEXT BOOKS:

1. Tirupathi.R. Chandrupatla and Ashok D. Belegundu, Introduction to Finite elements in Engineering. PHI.
2. S Senthil, Introduction of Finite Element Analysis. Laxmi Publications.
3. SMD Jalaluddin, Introduction of Finite Element Analysis.Anuradha Publications.
4. The Finite Element Method for Engineers – Kenneth H. Huebner, Donald John Wiley & sons (ASIA) Pte Ltd.

REFERENCES:

1. K. J. Bathe, Finite element procedures. PHI.
2. SS Rao, The finite element method in engineering. Butterworth Heinemann.
3. J.N. Reddy, An introduction to the Finite element method. TMH.
4. Chennakesava, R Alavala, Finite element methods: Basic concepts and applications. PHI.
5. K. J. Bathe, Finite element procedures. PHI. 6. SS Rao, The finite element method in engineering. Butterworth Heinemann.

COURSE OUTCOMES: Upon completion of this course, the students will be able to:

1. Describe the concept of FEM and difference between the FEM with other methods and problems based on 1-D bar elements and shape functions.
2. Derive elemental properties and shape functions for truss and beam elements and related problems.
3. Understand the concept deriving the elemental matrix and solving the basic problems of CST and axi-symmetric solids
4. Formulate FE characteristic equations for iso-parametric problems and Explore the concept of steady state heat transfer in fin and composite slab
5. Understand the concept of consistent and lumped mass models and solve the dynamic analysis of all types of elements.

Lecturer Notes

COURSE COVERAGE SUMMARY

Units	Chapter No's In The Text Book Covered	Author	Text Book Title	Publishers	Edition
Unit-I Introduction to FEM and One dimesional Elements	1&2	SMD Jalaluddin	Introduction of Finite Element Analysis	Anuradha Publications	4
Unit-II Trusses & Beams	3 &4	SMD Jalaluddin	Introduction of Finite Element Analysis	Anuradha Publications	4
Unit-III Two dimensional Problems &Axi-symmetric Models	12	SMD Jalaluddin	Introduction of Finite Element Analysis	Anuradha Publications	4
Unit-IV Iso-Parametric Formulation & Heat Transfer Problems	7&13	SMD Jalaluddin	Introduction of Finite Element Analysis	Anuradha Publications	4
Unit-V Dynamic Analysis	14	S M Jalaud Ddin	Introduction of Finite Element Analysis	Anuradha Publications	4

UNIT 1

INTRODUCTION TO FEM

& ONE DIMENSIONAL PROBLEMS

Syllabus

Introduction to Finite Element Method for solving field problems. Stress and Equilibrium. Strain – Displacement relations. Stress – strain relations. One Dimensional problem: Finite element modeling, local coordinates and shape functions. Potential Energy approach, Assembly of Global stiffness matrix and load vector. Finite element equations, Treatment of boundary conditions, Quadratic shape functions and its applications.

OBJECTIVE:

To enable the students to understand fundamentals of finite element analysis and the principle's involved in the discretization of domain with various elements, polynomial interpolation and assembly of global arrays. .

OUTCOME:

Identify mathematical model to solve common engineering problems by applying the finite element method and formulate the elements for one dimensional bar structures and solve problems in one dimensional bar structures.

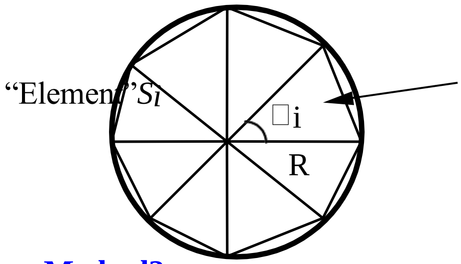
UNIT I
INTRODUCTION

Basic Concepts

The *finite element method* (FEM), or *finite element analysis* (FEA), is based on the idea of building a complicated object with simple blocks, or, dividing a complicated object into small and manage able pieces. Application of this simple idea can be found everywhere in everyday life, as well as in engineering.

Examples:

- Lego(kid's play)
- Buildings
- Approximation of the area of a circle:



Why Finite Element Method?

- Design analysis: hand calculations, experiments, and computer simulations
- FEM/FEA is the most widely applied computer simulation method in engineering
- Closely integrated with CAD/CAM applications

A Brief History of the FEM

- 1943 ----- Courant(Variational methods)
- 1956 ----- Turner, Clough, Martin and Topp(Stiffness)
- 1960 ----- Clough("Finite Element", plane problems)
- 1970s ------ Applications on main frame computers
- 1980s ------ Micro computers, pre-and post processors
- 1990s--------- Analysis of large structural systems

What is FEM?

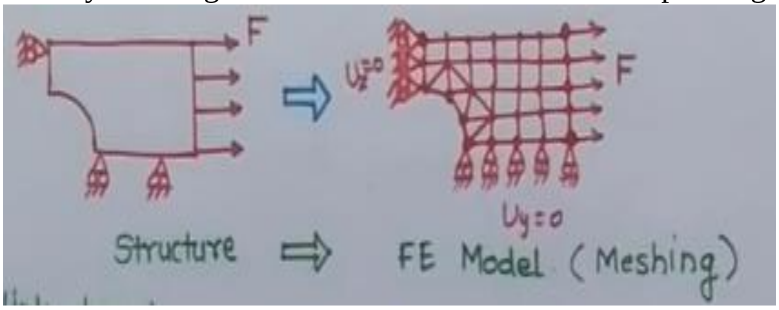
- Many physical phenomena a in engineering and science can be described in terms of *partial differential equations*.
- In general, solving these equations by classical analytical methods for arbitrary shapes is almost impossible.
- The *finite element method* (FEM) is a numerical approach by which these partial differential equations can be solved approximately.
- From an engineering standpoint, the FEM is a method for solving engineering problems such as *stress analysis, heat transfer, fluid flow* and *electromagnetic* by computer simulation.
- Millions of engineers and scientists worldwide use the FEM to predict the behavior of structural, mechanical, thermal, electrical and chemical systems for both design and performance analyses.

BASIC STEPS OF FEM

1 Discretization of the structure
2 Identify primary unknown quantity
3 Selection of Displacement function
4 Formation of the element stiffness matrix and load vector
5 Formation of Global stiffness matrix and load vector
6 In corporation of Boundary conditions
7 Solution of Simultaneous equations
8 Calculation of element strains and stresses
9 Interpretation of the result obtained.

Step1: Discretization of or structure–(Establish the FE mesh)

- The continuum is divided into a number of elements by imaginary lines or surfaces.
- The inter connected elements may have different sizes and shapes.
- Establish the FE mesh with set coordinates, element numbers and node numbers
- The discretized FE model must be situated with a coordinate system
- Elements and nodes in the discretized FE model need to be identified by "element numbers"and"nodal numbers."
- Nodes are identified by the assigned node numbers and their corresponding coordinates



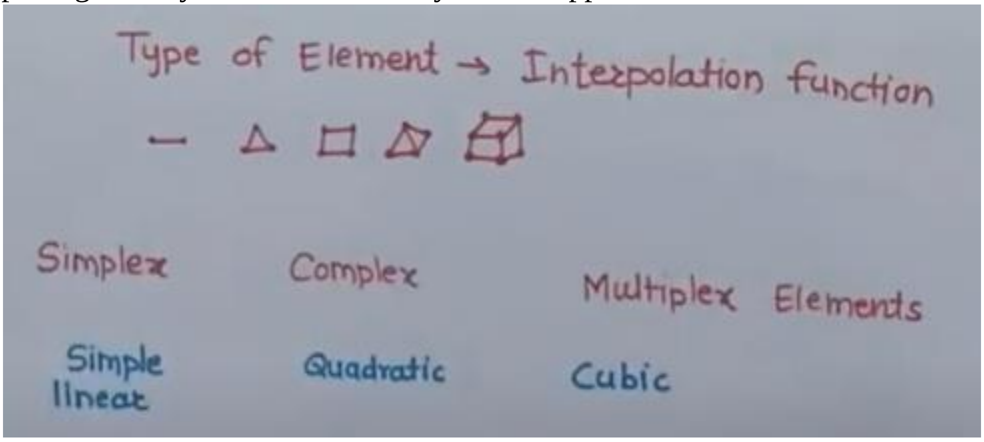
Step2:Identifyprimaryunknownquantity

- Primary unknown quantity - The first and principal unknown quantity to be obtained by the FEM

  Eg: Stress analysis: Displacement{u}at nodes
- In stress analysis, The primary unknowns are nodal displacements, but secondary unknown quantities include: strains in elements can be obtained by the"strain-displacementrelations,"andtheunknownstressesintheelements by the stress-strain relations (the Hooke's law).

Step3:Choice of approximating functions

- Displacement function is the starting point of the mathematical analysis.
- This represents the variation of the displacement with in the element.
- Thedisplacementfunctionmaybeapproximatedintheformalinearfunction or a higher-order function.
- A convenient way to express it is by polynomial expressions.
- Shape or geometry of the element may also be approximated.



Step4:Formation of the element stiffness matrix & load vector

- After continuum is discretized with desired element shapes, the in dividual element stiffness matrix is formulated.
- Basically it is a minimization procedure what ever may be the approach adopted.
- For certain elements, the form involves a great deal of sophistication.
- The geometry of the element is defined in reference to the global frame.
- Coordinate transformation must be done for elements where it is necessary.

$$\{F\}e=\{K\}e^*\{q\}e$$

Step5:Formation of over all stiffness matrix & load vector

- After the element stiffness matrices in global coordinates are formed, they are assembled to form the overall stiffness matrix.
- The assembly is done through the nodes which are common to adjacent elements.
- The over all stiffness matrix is symmetric and banded.

$$\{F\}G=\{K\}G^*\{q\}G$$

Step6:Incorporation of boundary conditions

- The boundary restraint conditions are to be imposed in the stiffness matrix.
- There are various techniques available to satisfy the boundary conditions.
- One is the size of the stiffness matrix may be reduced or condensed in its final

form.

- To ease computer programming aspect and to elegantly incorporate the boundary conditions, the size of overall matrix is kept the same.

Step7:Solution of simultaneous equations

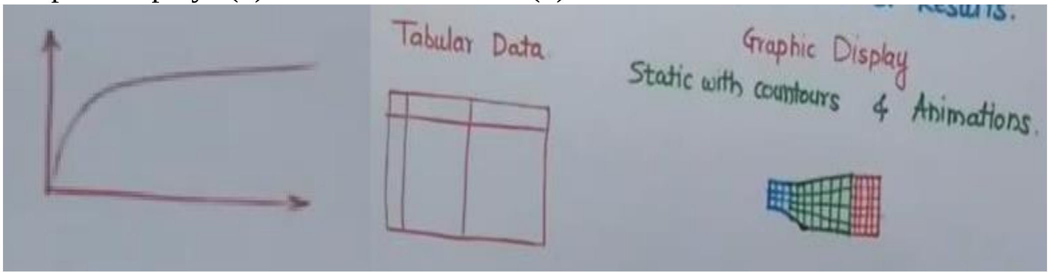
- The unknown nodal displacements are calculated by the multiplication of force vector with the inverse of stiffness matrix.
- [δ]=inverse of [k].[F]

Step8:Calculation of stresses or stress-resultants

- Nodal displacements are utilized for the calculation of stresses or stress- resultants.
- This may be done for all elements of the continuum or it may be limited to some predetermined elements.

Step9:Display and Interpretation of Results

- Results may also be obtained by graphical means.
- It may desirable to plot the contours of the deformed shape of the continuum.
- Tabulation of results
- Graphic displays:(1)Static with contours.(2)Animations



Advantages of Finite Element Method

- Modeling of complex geometries and irregular shapes are easier as varieties of finite elements are available for Discretization of domain.
- Boundary conditions can be easily incorporated in FEM.
- Different types of material properties can be easily accommodated in modeling from element to element or even within an element.
- Higher order elements may be implemented.
- FEM is simple, compact and result-oriented and hence widely popular among engineering community.
- Availability of large number of computer software packages and literature makes FEM a versatile and powerful numerical method.

Disadvantages of Finite Element Method

- Large amount of data is required as input for the mesh used in terms of nodal connectivity and other parameters depending on the problem.
- It requires a digital computer and fairly extensive
- It requires longer execution time compared with FEM.
- Out put result will vary considerably.

Limitations of FEA

1. Proper engineering judgment is to be exercised to interpret results.
2. It requires large computer memory and computational time to obtain in tend results.
3. There are certain categories of problems where other methods are more effective, e.g., fluid problems having boundaries at infinity are better treated by the boundary element method.
4. For some problems, there may be a considerable amount of input data. Errors may creep up in their preparation and the results thus obtained may also appear to be acceptable which indicates deceptive state of affairs. It is always desirable to make a visual check of the input data.
5. In the FEM, many problems lead to round-off errors. Computer works with alimitednumberofdigitsandsolvingtheproblemwithrestrictednumberofdigits may not yield the desired degree of accuracy or it may give total erroneous results in somecases.Formanyproblemstheincreaseinthenumberofdigitsforthepurpose of calculation improves the accuracy.

Applications of FEM

1. Mechanical engineering: In mechanical engineering, FEM applications include steady and transient thermal analysis in solids and fluids, stress analysis in solids, automotive design and analysis and manufacturing process simulation.

2. Geotechnical engineering: FEM applications include stress analysis, slope stability analysis, soil structure interactions, seepage of fluids in soils and rocks, analysis of dams, tunnels, bore holes, propagation of stress waves and dynamic soil structure interaction.

3. Aerospace engineering: FEM is used for several purposes such as structural analysis for natural frequencies, modes shapes, response analysis and aerodynamics.

4. Nuclear engineering: FEM applications include steady and dynamic analysis of reactor containment structures, thermo-viscoelastic analysis of reactor components, steady and transient temperature-distribution analysis of reactors and related structures.

5. Electrical and electronics engineering: FEM applications include electrical network analysis, electromagnetic, insulation design analysis in high-voltage equipments, dynamic analysis of motors and heat analysis in electrical and electronic equipments.

6. Metallurgical, chemical engineering: In metallurgical engineering, FEM is used for the metallurgical process simulation, moulding and casting. In chemical engineering, FEM can be used in the simulation of chemical processes, transport processes and chemical reaction simulations.

7. Meteorology and bio-engineering: In the recent times, FEM is used in climate predictions, monsoon prediction and wind predictions. FEM is also used in bio-engineering for the simulation of various human organs, blood circulation prediction and even total synthesis of human body.

8 Civil Engineering Structure: Finite element analysis (FEA) is an extremely useful tool in the field of civil engineering for numerically approximating physical structures that are too complex for regular analytical solutions. Consider a concrete beam with support at both ends, facing a concentrated load on its center span. The deflection at the center span can be determined mathematically in a relatively simple way, as the initial and boundary conditions are finite and in control. However, once you transport the same beam into a practical application, such as within a bridge, the forces at play become much more difficult to analyze with simple mathematics.

finite element method vs *classical method*

Classical Methods	Finite Element method
1) Exact equations are formed and exact solutions are obtained.	1) Exact equations are formed but approximate solutions are formed.
2) Solutions can be obtained for few standard cases.	2) Solution can be obtained for all problems.
3) For the solution of shape, Boundary conditions and loading some assumptions are made.	3) No assumptions are made problem is treated as it is.
4) When material is not isotropic, solution for the problems becomes very difficult.	4) All type of property can handle without any difficulty.
5) If structure consist more than one material, it is difficult to analyze.	6) If structure consist more than one material then it can be analyzed without any difficulty.

POTENTIAL ENERGY AND EQUILIBRIUM; THE RAYLEIGH–RITZ METHOD

In mechanics of solids, our problem is to determine the displacement **u** of the body shown in Fig. 1.1, satisfying the equilibrium equations 1.6. Note that stresses are related to strains, which, in turn, are related to displacements. This leads to requiring solution of second-order partial differential equations. Solution of this set of equations is generally referred to as an *exact* solution. Such exact solutions are available for simple geometries and loading conditions, and one may refer to publications in theory of elasticity. For problems of complex geometries and general boundary and loading conditions, obtaining such solutions is an almost impossible task. Approximate solution methods usually employ potential energy or variational methods, which place less stringent conditions on the functions.

Potential Energy, Π

The total potential energy Π of an elastic body, is defined as the sum of total strain energy (U) and the work potential:

$$\Pi = \underset{(U)}{\text{Strain energy}} + \underset{(\text{WP})}{\text{Work potential}} \tag{1.24}$$

For linear elastic materials, the strain energy per unit volume in the body is $\frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\epsilon}$. For the elastic body shown in Fig. 1.1, the total strain energy U is given by

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma}^T\boldsymbol{\epsilon}\,dV \tag{1.25}$$

The work potential WP is given by

$$\text{WP} = -\int_V \mathbf{u}^T\mathbf{f}\,dV - \int_S \mathbf{u}^T\mathbf{T}\,dS - \sum_i \mathbf{u}_i^T\mathbf{P}_i \tag{1.26}$$

The total potential for the general elastic body shown in Fig. 1.1 is

Principle of Minimum Potential Energy

For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.

Example 1.2

The potential energy for the linear elastic one-dimensional rod (Fig. E1.2), with body forc neglected, is

$$\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - 2u_1$$

where $u_1 = u(x = 1)$.

Let us consider a polynomial function

$$u = a_1 + a_2x + a_3x^2$$

This must satisfy $u = 0$ at $x = 0$ and $u = 0$ at $x = 2$. Thus,

$$0 = a_1$$
$$0 = a_1 + 2a_2 + 4a_3$$

Hence,

$$a_2 = -2a_3$$
$$u = a_3(-2x + x^2) \qquad u_1 = -a_3$$

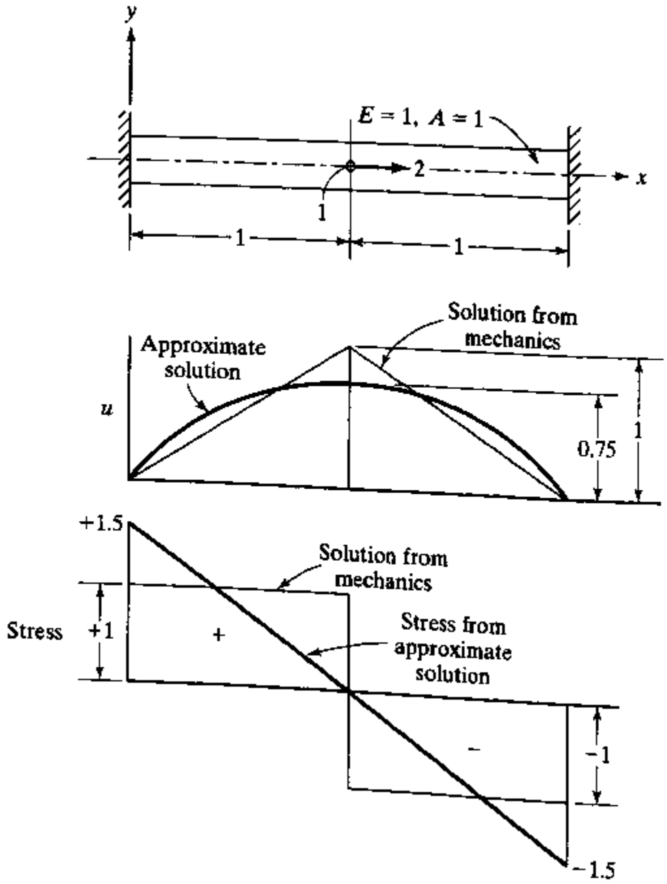


FIGURE E1.2

Then $du/dx = 2a_3(-1 + x)$ and

$$\Pi = \frac{1}{2}\int_0^2 4a_3^2(-1 + x)^2\,dx - 2(-a_3)$$

$$= 2a_3^2 \int_0^2 (1 - 2x + x^2)\,dx + 2a_3$$

$$= 2a_3^2\left(\tfrac{2}{3}\right) + 2a_3$$

We set $\partial\Pi/\partial a_3 = 4a_3\left(\tfrac{2}{3}\right) + 2 = 0$, resulting in

$$a_3 = -0.75 \qquad u_1 = -a_3 = 0.75$$

The stress in the bar is given by

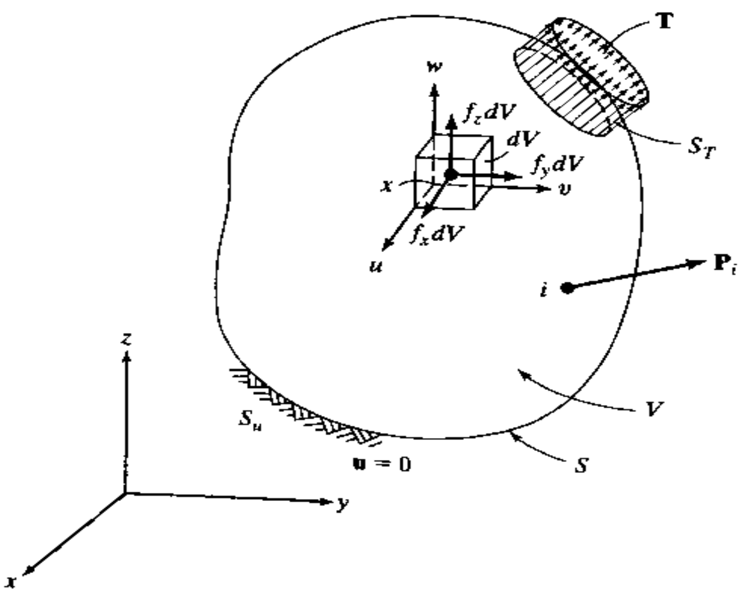
$$\sigma = E\frac{du}{dx} = 1.5(1 - x)$$ ■

We note here that an exact solution is obtained if piecewise polynomial interpolation is used in the construction of u.

The finite element method provides a systematic way of constructing the basis functions ϕ_i used in Eq. 1.30.

STRESSES AND EQUILIBRIUM

A three-dimensional body occupying a volume V and having a surface S is shown in Fig. 1.1. Points in the body are located by x, y, z coordinates. The boundary is constrained on some region, where displacement is specified. On part of the boundary, dis-



tributed force per unit area **T**, also called traction, is applied. Under the force, the body deforms. The deformation of a point **x** ($= [x, y, z]^{\mathrm{T}}$) is given by the three components of its displacement:

$$\mathbf{u} = [u, v, w]^{\mathrm{T}} \tag{1.1}$$

The distributed force per unit volume, for example, the weight per unit volume, is the vector **f** given by

$$\mathbf{f} = [f_x, f_y, f_z]^{\mathrm{T}} \tag{1.2}$$

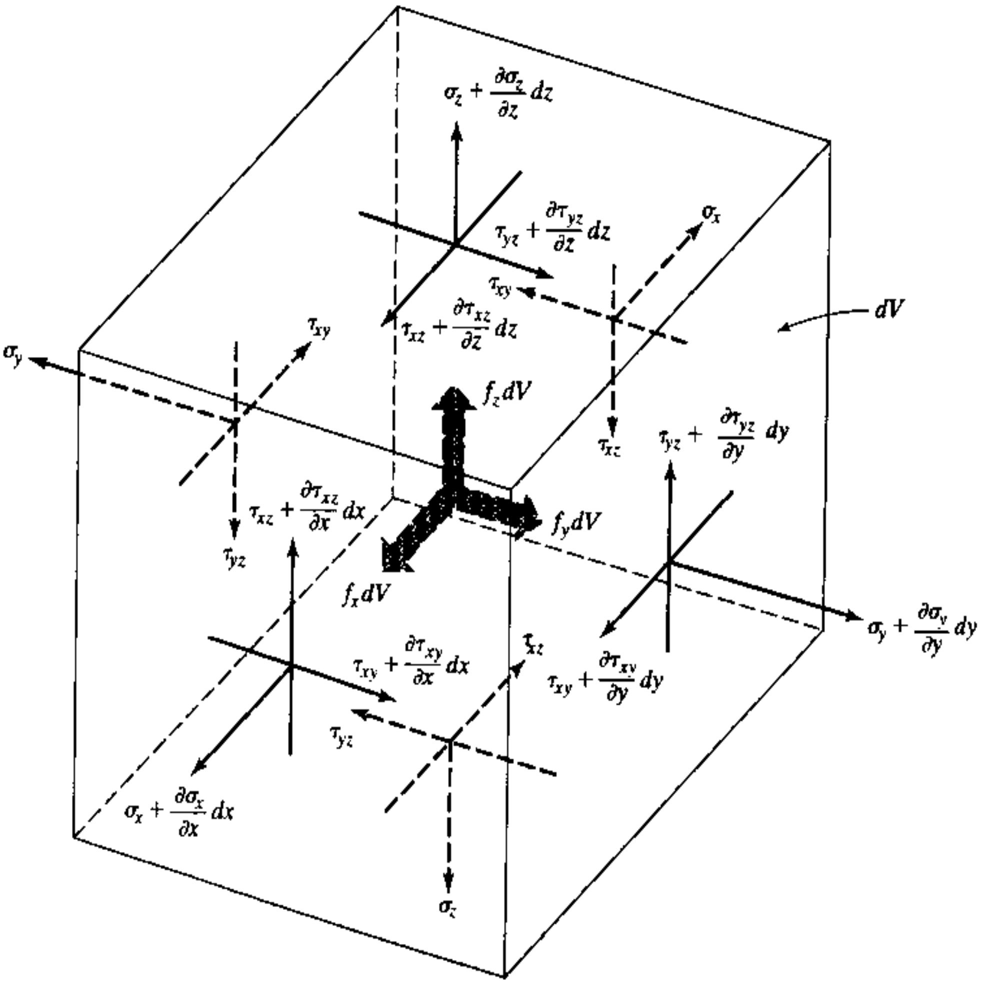
The body force acting on the elemental volume dV is shown in Fig. 1.1. The surface traction **T** may be given by its component values at points on the surface:

$$\mathbf{T} = [T_x, T_y, T_z]^{\mathrm{T}} \tag{1.3}$$

Examples of traction are distributed contact force and action of pressure. A load **P** acting at a point i is represented by its three components:

$$\mathbf{P}_i = [P_x, P_y, P_z]_i^{\mathrm{T}} \tag{1.4}$$

The stresses acting on the elemental volume dV are shown in Fig. 1.2. When the volume dV shrinks to a point, the stress tensor is represented by placing its components in a



(3 × 3) symmetric matrix. However, we represent stress by the six independent components as in

$$\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}]^T \tag{1.5}$$

where σ_x, σ_y, σ_z are normal stresses and τ_{yz}, τ_{xz}, τ_{xy}, are shear stresses. Let us consider equilibrium of the elemental volume shown in Fig. 1.2. First we get forces on faces by multiplying the stresses by the corresponding areas. Writing $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma F_z = 0$ and recognizing $dV = dx\,dy\,dz$, we get the equilibrium equations

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0
\end{aligned} \tag{1.6}$$

Special Cases

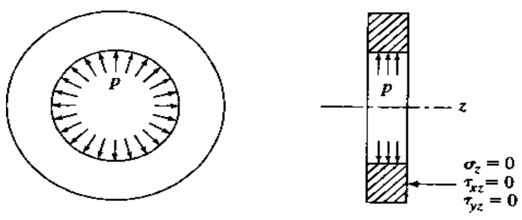
One dimension. In one dimension, we have normal stress σ along x and the corresponding normal strain ϵ. Stress–strain relations (Eq. 1.14) are simply

$$\sigma = E\epsilon \tag{1.16}$$

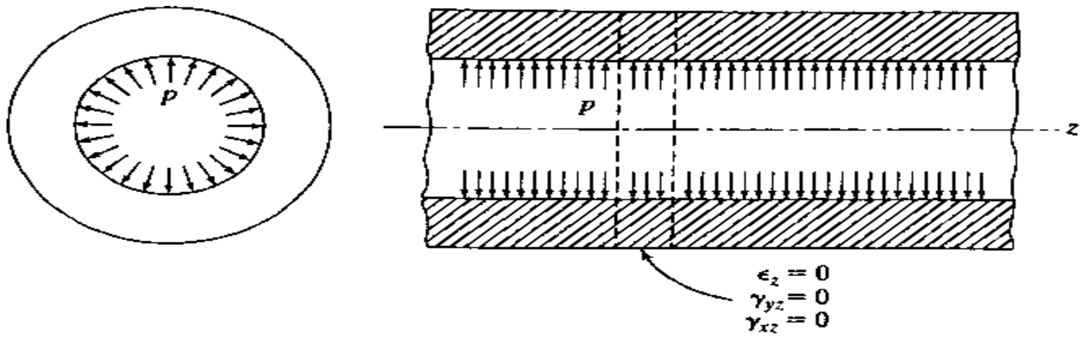
Two dimensions. In two dimensions, the problems are modeled as plane stress and plane strain.

Plane Stress. A thin planar body subjected to in-plane loading on its edge surface is said to be in plane stress. A ring press fitted on a shaft, Fig. 1.5a, is an example. Here stresses σ_z, τ_{xz}, and τ_{yz} are set as zero. The Hooke's law relations (Eq. 1.11) then give us

$$\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \\
\epsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} \\
\gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy} \\
\epsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y)
\end{aligned} \tag{1.17}$$



(a)



(b)

FIGURE 1.5 (a) Plane stress and (b) plane strain.

The inverse relations are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (1.18)$$

which is used as $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$.

Plane Strain. If a long body of uniform cross section is subjected to transverse loading along its length, a small thickness in the loaded area, as shown in Fig. 1.5b, can be treated as subjected to plane strain. Here ϵ_z, γ_{zx}, γ_{yz} are taken as zero. Stress σ_z may not be zero in this case. The stress–strain relations can be obtained directly from Eqs. 1.14 and 1.15:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (1.19)$$

$\mathbf{D}$ here is a (3×3) matrix, which relates three stresses and three strains.

Anisotropic bodies, with uniform orientation, can be considered by using the appropriate $\mathbf{D}$ matrix for the material.

Strain Displacement Relationship for Axi-symmentric element:

Consider an axi-symmetric ring element and its cross section to represent the general state of strain for an axi-symmetric problem. The displacements can be expressed for element *ABCD* in the plane of a cross-section in cylindrical coordinates. We then let *u* and *w* denote the displacements in the radial and longitudinal directions, respectively. The side *AB* of the element is displaced an amount *u* ,and side *CD* is then displaced an amount *u* +in the radial direction.

The strain in the tangential direction depends on the tangential displacement v and on the radial displacement u.
However, for axisymmetric deformation behavior, recall that the tangential displacement v is equal to zero.

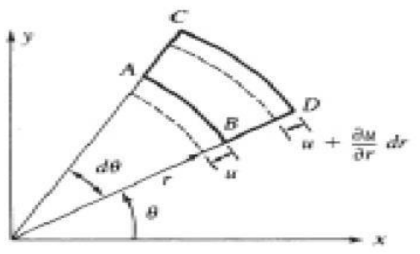
The tangential strain is due only to the radial displacement.

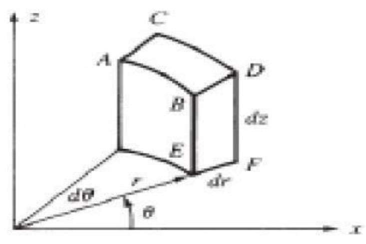
Having only radial displacement u, the new length of the arc AB is $(r + u)d\theta$, and the tangential strain is then given by:

$$\varepsilon_\theta = \frac{(r+u)d\theta - rd\theta}{rd\theta} = \frac{u}{r}$$

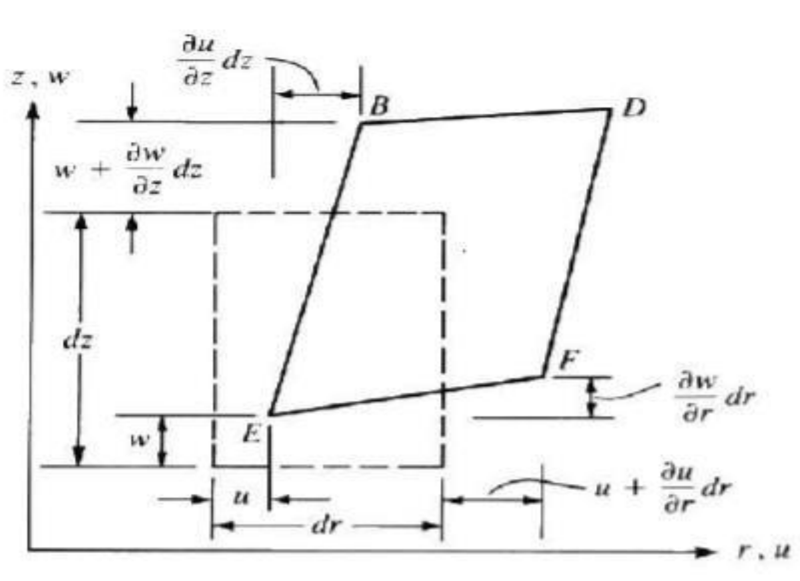
Consider the longitudinal element BDEF to obtain the longitudinal strain and the shear strain. The element displaces by amounts u and w in the radial and longitudinal directions at point E.

The element displaces additional amounts: $(\partial w/\partial z)dz$ along line BE and $(\partial u/\partial r)dr$ along line EF.





The normal strain in the radial direction is then given by: $\varepsilon_r = \frac{\partial u}{\partial r}$



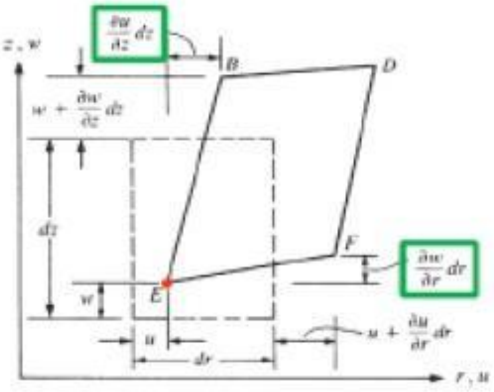
Furthermore, observing lines EF and BE, we see that point F moves upward an amount $(\partial w/\partial r)dr$ with respect to point E and point B moves to the right an amount $(\partial u/\partial z)dz$ with respect to point E.

The longitudinal normal strain is given by:

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

The shear strain in the *r-z* plane is:

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$



Summarizing the strain-displacement relationships gives:

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

COORDINATES AND SHAPE FUNCTIONS

Consider a typical finite element e in Fig. 3.5a. In the local number scheme, the first node will be numbered 1 and the second node 2. The notation x_1 = x-coordinate of node 1, x_2 = x-coordinate of node 2 is used. We define a **natural** or **intrinsic** coordinate system, denoted by ξ, as

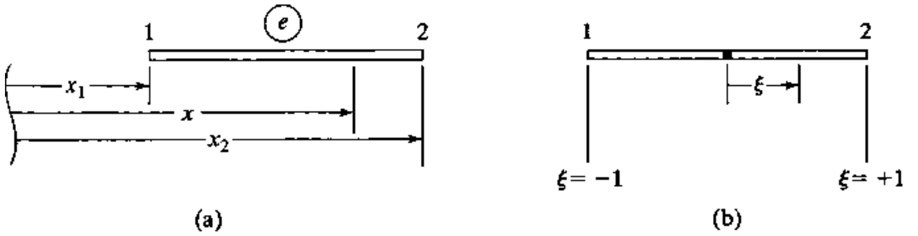


FIGURE 3.5 Typical element in x- and ξ-coordinates.

$$\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1 \tag{3.4}$$

From Fig. 3.5b, we see that $\xi = -1$ at node 1 and $\xi = 1$ at node 2. The length of an element is covered when ξ changes from -1 to 1. We use this system of coordinates in defining shape functions, which are used in interpolating the displacement field.

Now *the unknown displacement field within an element will be interpolated by a linear distribution* (Fig. 3.6). This approximation becomes increasingly accurate as more elements are considered in the model. To implement this linear interpolation, linear **shape functions** will be introduced as

$$N_1(\xi) = \frac{1 - \xi}{2} \tag{3.5}$$

$$N_2(\xi) = \frac{1 + \xi}{2} \tag{3.6}$$

The shape functions N_1 and N_2 are shown in Figs. 3.7a and b, respectively. The graph of the shape function N_1 in Fig. 3.7a is obtained from Eq. 3.5 by noting that $N_1 = 1$ at $\xi = -1$, $N_1 = 0$ at $\xi = 1$, and N_1 is a straight line between the two points. Similarly, the graph of N_2 in Fig. 3.7b is obtained from Eq. 3.6. Once the shape functions are defined, the linear displacement field within the element can be written in terms of the nodal displacements q_1 and q_2 as

$$u = N_1 q_1 + N_2 q_2 \tag{3.7a}$$

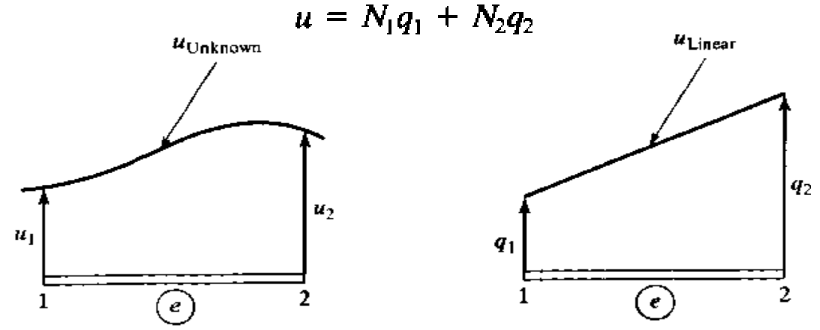


FIGURE 3.6 Linear interpolation of the displacement field within an element.

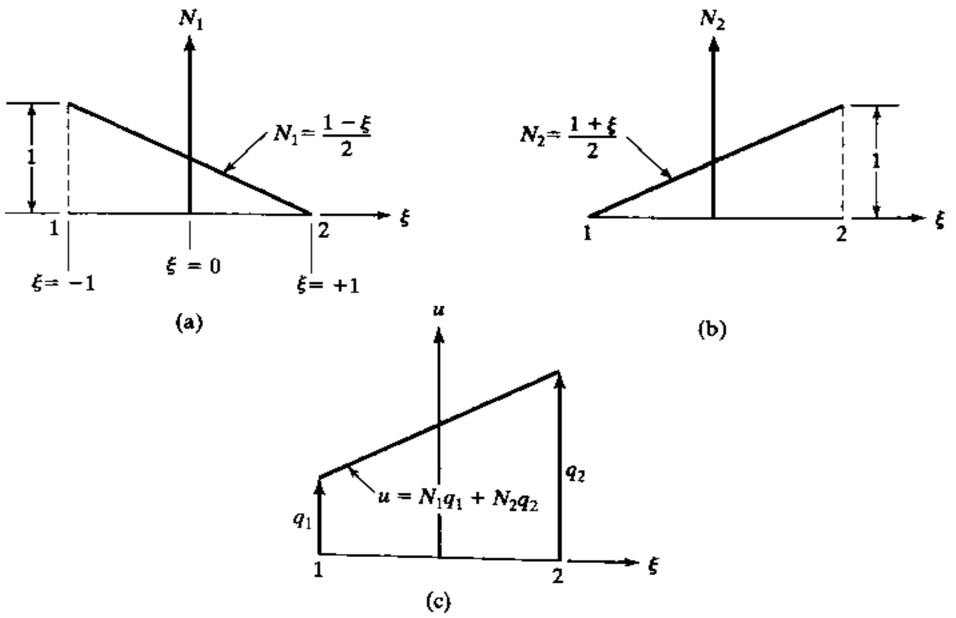


FIGURE 3.7 (a) Shape function N_1, (b) shape function N_2, and (c) linear interpolation using N_1 and N_2.

or, in matrix notation, as

$$u = \mathbf{Nq} \tag{3.7b}$$

where

$$\mathbf{N} = [N_1, N_2] \quad \text{and} \quad \mathbf{q} = [q_1, q_2]^{\mathrm{T}} \tag{3.8}$$

In these equations, $\mathbf{q}$ is referred to as the *element displacement vector*. It is readily verified from Eq. 3.7a that $u = q_1$ at node 1, $u = q_2$ at node 2, and that u varies linearly (Fig. 3.7c).

It may be noted that the transformation from x to ξ in Eq. 3.4 can be written in terms of N_1 and N_2 as

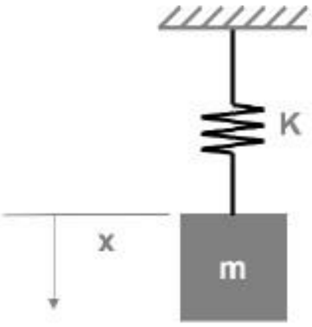
$$x = N_1x_1 + N_2x_2 \tag{3.9}$$

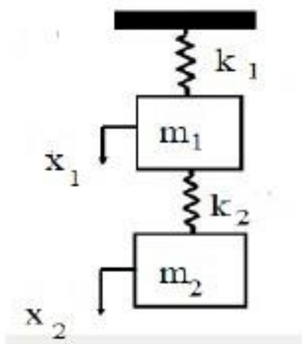
Comparing Eqs. 3.7a and 3.9, we see that both the displacement u and the coordinate x are interpolated within the element using the *same* shape functions N_1 and N_2. This is referred to as the *isoparametric* formulation in the literature.

Though linear shape functions have been used previously, other choices are possible. Quadratic shape functions are discussed in Section 3.9. In general, shape functions need to satisfy the following:

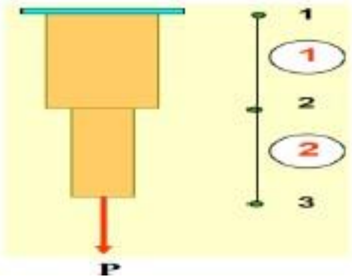
Degrees of Freedom in FEA:

- Degree of Freedom (DoF) is a "possibility" to move in a defined direction. There are 6 DoF in a 3D space: you can move or rotate along axis x, y or z. Together, those components describe a motion in 3D. DoF in FEA also do other things : they control supports, information about stresses and more.
- Degree of freedom or DOF means the number of independent coordinates a structure can move. There are 6 DOF possible for a structure. They are movement onx,y and z axis and rotation about these axis.
- What ever be the field, degree of freedom, do fin short, represents the minimum number of independent coordinates required to specify the position of every mass in the system uniquely.
- eg.A simple spring mass system as shown in Fig. which is constrained to move only in the vertical direction requires the displacement x only to specify the position of the mass m. Hence it has one degree of freedom.
- If we attach another spring and another mass below the first mass then each masswillundergodifferentdisplacementandhenceweneedtospecifyx1 and x2 which are the displacements of masses 1 and 2 respectively. Hence this has 2 dof.

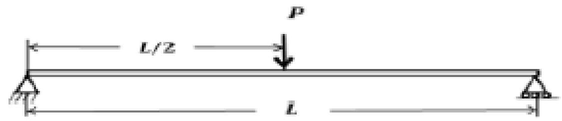


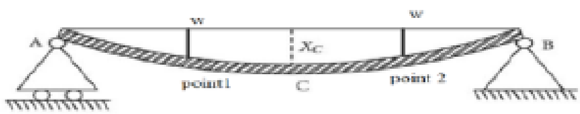


- Now coming to FEM when we want to find the stresses in a member subjected to axial loads such as the stepped bar shown below, since the bar is long and thin we can assume that the longitudinal displacements are significantly higher than the lateral displacements. So neglecting this lateral displacement we can discretise this system into two 2 noded elements.

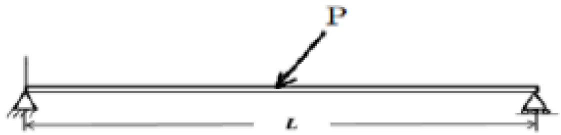


- When we use a 2 noded element it is assumed that there is only on degree of freedom at each node namely the axial displacement of that node. So total degrees of freedom in this case for one element is 2.
- On the other hand in a beam element that is subjected to only vertical transverse loads, we require minimum 2 dofs. Why is this so?
- Take the case of a simply supported beam subjected to a central point load as shown in the figure below.





- If we specify the position of every point in the beam with only one variable namely the transverse displacement w, then if we look for two symmetrically placed points along the beam such as points 1 and 2, the displacements will be the same and equal to w. So if have to specifically refer to only one point uniquely we need one more variable that can be used to identify that point. Hence we introduce another variable namely the slope of the deflection curve.
- So a simple beam subjected to only vertical loads can be modelled using a beam element that has 2 dofs per node namely So total dof s for one two noded beam element is 2 x 2 =4.
- If the beam is subjected to a load as shown below



- Then there is an axial displacement that comes into the picture additionally. So we have to introduce one more dof namely axial displacement u at each node thus bringing the dof per node to 3 and total dof to 6.
- Similarly a 3 noded triangular element used to model a thin rectangular fin has one dof (variable) per node namely temperature so total dof is 3 x 1 =3. In a structural application there will be two dof per node namely u and v displacements. Hence total dof for a 3 noded triangular element for stress analysis will be 3 x 2 =6.
- A 4 noded tetrahedral solid element has 3 dof per node (u,v,and v displacements) when used in structural applications so total dof is 4 x 3 =12.
- So we need to understand the physical behaviour of the system and model it appropriately.

Shape Functions:

In the finite element analysis aim is to find the field variables at nodal points by rigorous analysis, assuming at any point inside the element basic variable is a function of values at nodal points of the element. This function which relates the field variable at any point within the element to the field variables of nodal points is called shape function. This is also called as interpolation function and approximating function. In two dimensional stress analysis in which basic field variable is displacement,

Shape functions are the polynomials meant to describe the variation of primary variable along the domain of element.

$$u = \Sigma N_i u_i , v = \Sigma N_i v_i \qquad ...(5.1)$$

where summation is over the number of nodes of the element. For example for three noded triangular element, displacement at $P(x, y)$ is

$$u = \Sigma N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = \Sigma N_i v_i = N_1 v_1 + N_2 v_2 + N_3 v_3$$

i.e.,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\underset{2\times 1}{\{\delta\}} = \underset{2\times 6}{[N]} \underset{6\times 1}{\{\delta\}_e}$$

where q is displacement at any point in the element

$[N]$ shape function

$\{\delta\}_e$ is vector of nodal displacements

Similarly in case of 6 noded triangular element

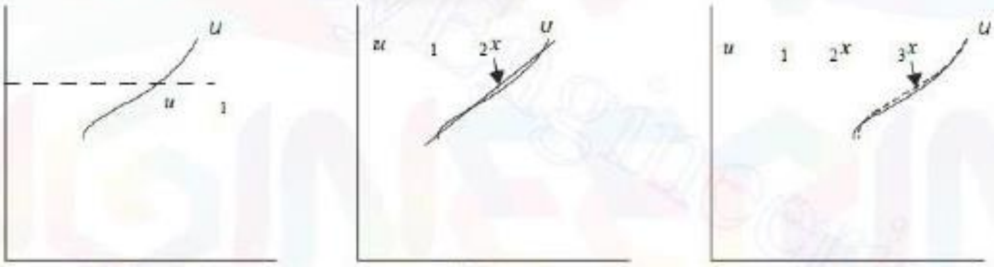
$$\underset{2\times 1}{\{\delta\}} = \underset{2\times 12}{[N]} \underset{12\times 1}{\{\delta\}_e}$$

POLYNOMIAL SHAPE FUNCTIONS

Polynomials are commonly used as shape functions. There are two reasons for using them:

(i) They are easy to handle mathematically i.e. differentiation and integration of polynomials is easy.

(ii) Using polynomial any function can be approximated reasonably well. If a function is highly nonlinear we may have to approximate with higher order polynomial. Fig. 5.1 shows approximation of a nonlinear one dimensional function by polynomials of different order.



(a) Constant (b) Linear (c) Quadratic

Approximation with polynomials

One Dimensional Polynomial Shape Function

A general one dimensional polynomial shape function of *n*th Order is given by,

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \dots \alpha_{n+1} x^n$$

In matrix form $u = [G]\{\alpha\}$

where $$[G] = [1, x, x^2 \dots x^n]$$

and $$\{\alpha\}^T = [\alpha_1\ \alpha_2\ \alpha_3 \dots \alpha_{n+1}]$$

Thus in one dimensional n^{th} order complete polynomial there are $m = n + 1$ terms.

Two Dimensional Polynomial Shape Function

A general form of two dimensional polynomial model is

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 \dots + \alpha_m y^n$$
$$v(x, y) = \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + \dots + \alpha_{2m} y^n \qquad \dots(5.6)$$

or $$\{\delta\} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = [G]\{\alpha\} = \begin{bmatrix} G_1 & 0 \\ 0 & G_1 \end{bmatrix}\{\alpha\} \qquad \dots(5.7)$$

where $$G_1 = [1\ x\ y\ x^2\ xy\ y^2\ x^3 \dots y^n]$$

$$\{\alpha\}^T = [\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4 \dots \alpha_{2m}]$$

It may be observed that in two dimensional problem, total number of terms m in a complete nth degree polynomial is

$$m = \frac{(n+1)(n+2)}{2} \qquad \dots(5.8)$$

For first order complete polynomial $n = 1$,

$$\therefore m = \frac{(1+1)(1+2)}{2} = 3$$

Another convenient way to remember complete two dimensional polynomial is in the form of Pascal Triangle shown in Fig. 5.2

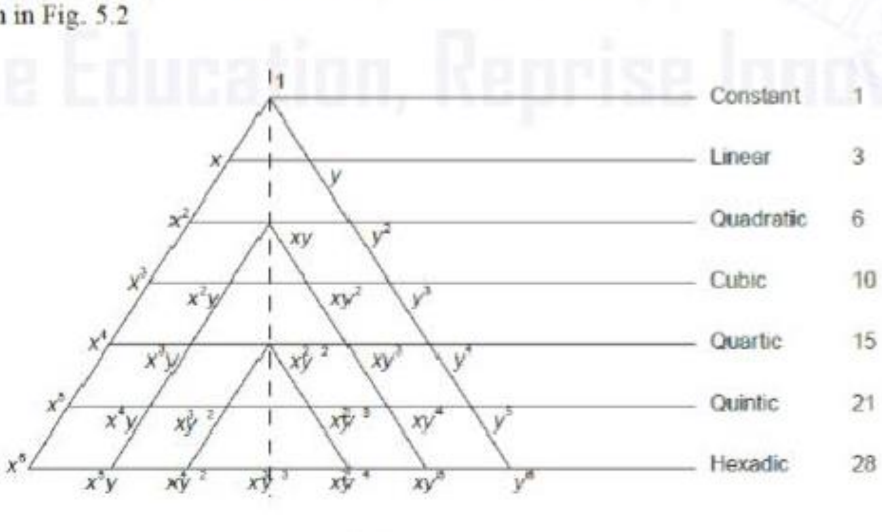


Fig. 5.2 *Pascal triangle*

Three Dimensional Polynomial Shape Function

A general three dimensional shape function of nth order complete polynomial is given by

$$\begin{aligned} u(x, y, z) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \dots + \alpha_m x^{n-1} z \\ v(x, y, z) &= \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + \alpha_{m+4} z + \alpha_{m+5} x^2 + \dots + \alpha_{2m} x^{n-1} z \\ w(x, y, z) &= \alpha_{2m+1} + \alpha_{2m+2} x + \alpha_{2m+3} y + \alpha_{2m+4} z + \dots + \alpha_{3m} x^{n-1} z \end{aligned} \quad \dots(5.9)$$

or

$$\delta(x, y, z) = \begin{Bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{Bmatrix} = [G]\{\alpha\} = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_1 & 0 \\ 0 & 0 & G_1 \end{bmatrix} \{\alpha\} \quad \dots(5.10)$$

Where $G_1 = [1\ x\ y\ z\ x^2\ xy\ y^2\ yz\ z^2\ zx\ \dots z^n\ z^{n-1}x\ \dots\ zx^{n-1}]$

and

$$\{\alpha\}^T = [\alpha_1\ \alpha_2\ \alpha_3\ \dots\ \alpha_{3m}]$$

It may be observed that a complete nth order polynomial in three dimensional case is having number of terms m given by the expression

$$m = \frac{(n+1)(n+2)(n+3)}{6}$$

Thus when $n = 1$, $m = \dfrac{(1+1)(1+2)(1+3)}{6} = 4$

i.e.

$$\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$$

5.3 CONVERGENCE REQUIREMENTS OF SHAPE FUNCTIONS

Numerical solutions are approximate solutions. Stiffness coefficients for a displacements model have higher magnitudes compared to those for the exact solutions. In other words the displacements obtained by finite element analysis are lesser than the exact values. Thus the FEM gives lower bound values. Hence it is desirable that as the finite element analysis mesh is refined, the solution approaches the exact values. This requirement is shown graphically in Fig. 5.4. In order to ensure this convergence criteria, the shape functions should satisfy the following requirement:

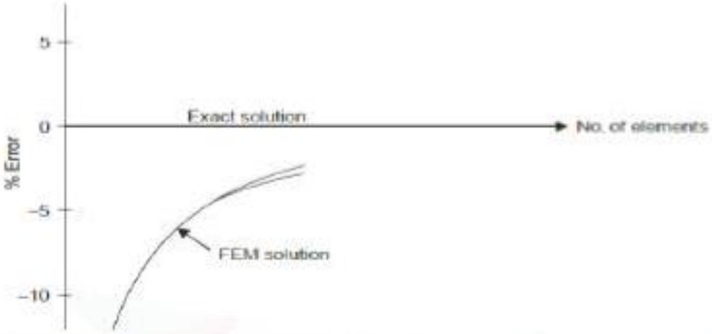


Fig. 5.4 *Convergence of FEM solution*

1. The displacement models must be continuous within the elements and the displacements must be compatible between the adjacent elements. The second part implies that the adjacent elements must deform without causing openings, overlaps or discontinuities between the elements. This requirement is called **'compatibility requirement'**.

   According to Felippa and Clough this requirement is satisfied, if the displacement and its partial derivatives upto one order less than the highest order derivative appearing in strain energy function is continuous. Hence in plane stress and plane strain problems, it is enough if continuity of displacement is satisfied, since strain energy function includes only first order derivatives of the displacement (SE = ½ stress × strain). It implies, it is enough if C^0 continuity is ensured in plane stress and plane strain problems. In case of flexure problems (beams, plates, shells) the strain energy terms include second derivatives of displacements $\left(\text{like } \frac{1}{2}\frac{M^2}{EI} \text{ where } M = -EI\frac{d^2 w}{dx^2} \right)$.

   Hence to satisfy compatibility requirement, not only displacement continuity but slope continuity (C^1 -continuity) should be satisfied. Hence in flexure problems displacements and their first derivatives are selected as nodal field variables.

2. The displacement model should include the **rigid body displacements** of the element. It means in displacement model there should be a term which permit all points on the element to experience the same displacement. It is obvious, if such term do not exists, shifting of the origin of the coordinate system will cause additional stresses and strains, which should not occur. In the displacement model,

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

   the term α_1 provides for the rigid body displacement. Hence to satisfy the requirement of rigid body displacement, there should be constant term in the shape function selected.

3. The displacement models must include the **constant strain state** of the element. This means, there should exist combination of values of polynomial terms that cause all points in the element to experience the same strain. One such combination should occur for each possible strain. The necessity of this requirement is understood physically, if we imagine the refinement of the mesh. As these elements approach infinitesimal size, the strains within the element approach constant values. Unless the shape function term includes these constant strain terms, we cannot hope to converge to a correct solution. In the displacement model,

$$v = \alpha_{m+1} + \alpha_{m+2}x + \alpha_{m+3}y + \alpha_{m+4}x^2 + \ldots + \alpha_{2m}y^n$$

α_2 and α_{m+2} provide for uniform strain ε_x,

α_3 and α_{m+3} provide for uniform strain ε_y

An additional consideration in the selection of polynomial shape function for the displacement model is that the pattern should be independent of the orientation of the local coordinate system. This property is known as **Geometric Isotropy, Spatial Isotropy** or **Geometric Invariance.** There are two simple guidelines to construct polynomial series with the desired property of isotropy:

1. Polynomial of order n that are complete, have geometric isotropy.
2. Polynomial of order n that are not complete, yet contain appropriate terms to preserve 'symmetry' have geometric isotropy. The simple test for this property is to interchange x and y in two dimensional problems or x, y, z in cyclic order in three dimensional problems and see that the total expression do not change. However the arbitrary constants may change.

   For example, we wish to construct a cubic polynomial expression for an element that has eight nodal values assigned to it. In this situation, we have to drop two terms from the complete cubic polynomial which contains 10 terms. To maintain geometric isotropy drop only terms that occur in symmetric pairs i.e. x^3, y^3 or x^2y, xy^2. Thus the acceptable eight term cubic polynomials shape function exhibiting geometric isotropy are

   $$\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 xy^2$$

   and $\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 y^3$

   $$\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 xy^2$$

   and $\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 y^3$

   In finite element analysis, the safest approach to reach correct solution is to pick the shape functions that satisfy all the requirements. For some problems, however, choosing shape functions that meet all the requirements may be difficult and may involve excessive numerical computations. For this reason some investigators have ventured to formulate shape functions for the elements that do not meet compatibility requirements. In some cases acceptable convergence has been obtained. Such elements are called '**non-conforming elements**'. The main disadvantage of using non-conforming elements is that we no longer know in advance that correct solution is reached.

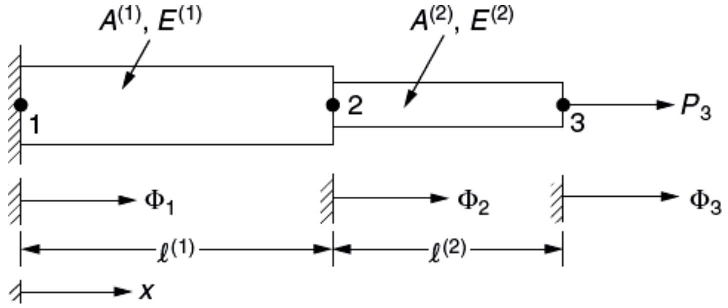
Characteristic of Shape function

1. Value of shape function of particular node is one and is zero to all other nodes.
2. Sum of all shape function is one.
3. Sum of the derivative of all the shape functions for a particular primary variable is zero.

Example 1.1: $A_1 = 200$ mm², $E_1 = E_2 = E = 2 \times 10^6$ N/mm²

$A_2 = 100$ mm², $l_1 = l_2 = 100$ mm

$P_3 = 1000$ N. Find: Displacement and stress & strain.



$$[K^{(1)}] = \frac{A^{(1)}E^{(1)}}{l^{(1)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \begin{matrix} \Phi_1 \\ \Phi_2 \end{matrix} \quad \text{(columns } \Phi_1, \Phi_2\text{)} \tag{E.16}$$

$$[K^{(2)}] = \frac{A^{(2)}E^{(2)}}{l^{(2)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{matrix} \Phi_2 \\ \Phi_3 \end{matrix} \quad \text{(columns } \Phi_2, \Phi_3\text{)} \tag{E.17}$$

Let overall stiffness matrix $[K] = [K^{(1)}] + [K^{(2)}]$

$$[\underset{\sim}{K}] = 10^6 \begin{bmatrix} 4 & -4 & 0 \\ -4 & 4+2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{matrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{matrix} = 2 \times 10^6 \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

In the present case, external loads act only at the node points; as such, there is no need to assemblethe element load vectors. The overall or global load vector can be written as

$$\vec{P} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 1 \end{Bmatrix}$$

$$2 \times 10^6 = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 1 \end{Bmatrix}$$

$$2 \times 10^6 \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

By solving the matrix

$\Phi_2 = 0.25 \times 10^{-6}$cm and

$\Phi_3 = 0.75 \times 10^{-6}$cm

Derive element strains and stresses.

Once the displacements are computed, the strains in the elements can be found as

$$\varepsilon^{(1)} = \frac{\partial \phi}{\partial x} \text{ for element } 1 = \frac{\Phi_2^{(1)} - \Phi_1^{(1)}}{l^{(1)}} \equiv \frac{\Phi_2 - \Phi_1}{l^{(1)}} = 0.25 \times 10^{-7}$$

$$\varepsilon^{(2)} = \frac{\partial \phi}{\partial x} \text{ for element } 2 = \frac{\Phi_2^{(2)} - \Phi_1^{(2)}}{l^{(2)}} \equiv \frac{\Phi_3 - \Phi_2}{l^{(2)}} = 0.50 \times 10^{-7}$$

The stresses in the elements are given by

$$\sigma^{(1)} = E^{(1)}\varepsilon^{(1)} = \left(2\times10^{7}\right)\left(0.25\times10^{-7}\right) = 0.5 \text{ N/cm}^2$$

$$\sigma^{(2)} = E^{(2)}\varepsilon^{(2)} = \left(2\times10^{7}\right)\left(0.50\times10^{-7}\right) = 1.0 \text{ N/cm}^2$$

Example 1.2:A thin plate as shown in Fig. (a) has uniform thickness of 2 cm and its modulus of elasticity is 200 x 10^3 N/mm^2 and density 7800 kg/m^3. In addition to its self-weight the plate is subjected to a point load P of500 N is applied at its midpoint.

Solve the following:

(i) Finite element model with two finite elements.

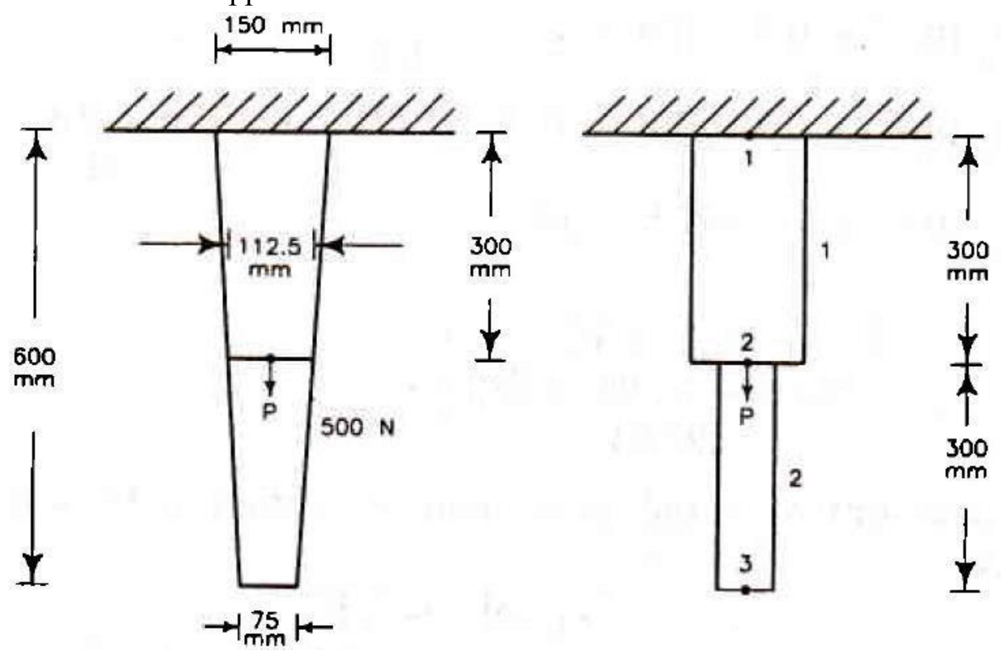
(ii) Global stiffness matrix.

(iii) Global load matrix.

(iv) Displacement at nodal point.

(v) Stresses in each element.

(vi) Reaction at support.



(a) (b)

(i) The tapered plate can be idealized as two element model with the tapered area converted to the rectangular equivalent area Refer Fig. (b). The areas A_1 and A_2

are equivalent areas calculated as

$$A_1 = \frac{15 + 11.25}{2} \times 2 = 26.25\ cm^2$$

$$A_2 = \frac{11.25 + 7.5}{2} \times 2 = 18.75\ cm^2$$

(ii) Global stiffness matrix can be obtained as

$$[k] = \frac{EA_1}{L_1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{EA_2}{L_2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{200 \times 10^3 \times 26.25 \times 10^2}{300}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \frac{200 \times 10^3 \times 18.75 \times 10^2}{300}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= 0.175 \times 10^7 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0.125 \times 10^7 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= 10^7 \begin{bmatrix} 0.175 & -0.175 & 0 \\ -0.175 & 0.3 & -0.125 \\ 0 & -0.125 & 0.125 \end{bmatrix}$$

(ii) The load matrix given by

$$F = \rho \begin{bmatrix} \frac{A_1 L_1}{2} \\ \frac{A_1 L_1}{2} + \frac{A_2 L_2}{2} \\ \frac{A_2 L_2}{2} \end{bmatrix} + \begin{bmatrix} -R_1 \\ P \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{26.25 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} - R_1 \\ \frac{26.25 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} + \frac{18.75 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} + P \\ \frac{18.75 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 30.75 - R_1 \\ 30.75 + 21.93 + 500 \\ 21.93 \end{bmatrix}$$

(iv) The displacement at nodal point can be obtained by writing the equation in global form as

$$[k]\ [\delta] = [F]$$

$$10^7 \begin{bmatrix} 0.175 & -0.175 & 0 \\ -0.175 & 0.3 & -0.125 \\ 0 & -0.125 & 0.125 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 30.75 - R_1 \\ 552.68 \\ 21.93 \end{bmatrix}.$$

Using elimination approach and eleminating first row and column in which reaction occurs.

$$10^7 \begin{bmatrix} 0.3 & -0.125 \\ -0.125 & 0.125 \end{bmatrix} \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 552.68 \\ 21.93 \end{bmatrix}$$

$$\delta_1 = 0, \quad \delta_2 = 3.28 \times 10^{-4} \text{ mm}, \ \delta_3 = 3.45 \times 10^{-4} \text{ mm}.$$

(v) The stress in the element 1

$$6_1 = \frac{E}{L_1} [-1, 1] \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \frac{200 \times 10^3}{300} \times 3.28 \times 10^{-4}$$

$$= 2.18 \times 10^{-1} \text{ MPa}$$

stress in the element 2

$$6_2 = \frac{E}{L_2} [-1, 1] \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix} = \frac{200 \times 10^3}{300} [-\delta_2 + \delta_3] = 0.11 \times 10^{-1} \text{ MPa}$$

(vi) The reaction node 1

$$R_1 = \frac{E A_1}{L_1} [\delta_2 - 30.75]$$

$$= 0.175 \times 10^7 \times 3.28 \times 10^{-4} - 30.75 = 543.25.$$

Penalty Approach :

- In the preceding problems, the elimination approach was used to achieve simplified matrices. This method though simple, is not very easy to adapt in terms of algorithms written fix computer programs.
- An alternate method to achieve solutions is by the penalty approach. By this approach a rigid support is considered as a spring having infinite stiffness. Consider a system as shown in Fig.

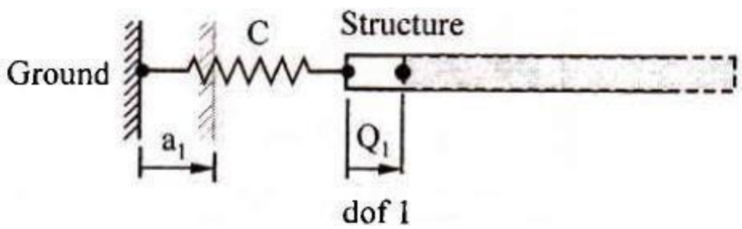


Fig. Penalty Approach

- The support or the ground is modelled with a high stiffness spring, having a stiffness C. To represent a rigid ground, c must be infinity.
- However, instead of introducing an infinite value in the calculations, a substantially high value of stiffness constant is introduced for those nodes resting on rigid supports.

- The magnitude of the stiffness constant should be at least 10^4 times more than the maximum value in the global stiffness matrix.
- From Fig. 1.6, it is seen that one end of the spring will displace by a_1. The displacement Q_1 (for dof 1) will be approximately equal to a_1 as the spring has a high stiffness.
- Consider a simple 1D element with node 1 fixed.

$$\mathbf{KQ} = \mathbf{F}$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

- At node 1, the stiffness term is „C" is introduced to reflect the boundary condition related to a rigid support. To compensate this change, the force term will also be modified as:

$$\begin{bmatrix} k_{11}+C & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 + Ca_1 \\ F_2 \end{Bmatrix}$$

- The reaction force as per penalty approach would be found by multiplying the added stiffness with the net deflection of the node.

$$R = -C(Q-a)$$

- The penalty approach is an approximate method and the accuracy of the forces depends on the value of C.

Example 1.3: Consider the bar shown in Fig.. An axial load P = 200 x 10^3 N is applied as shown.Using the penalty approach for handling boundary conditions, do the following:

(a) Determine the nodal displacements

(b) Determine the stress in each material.

(c) Determine the reaction forces.

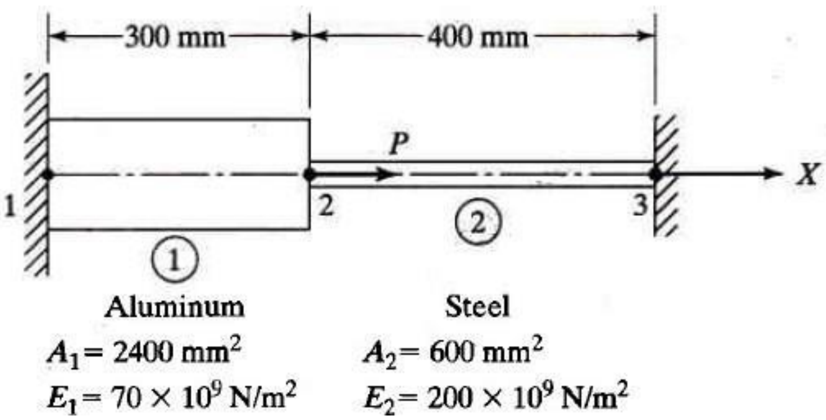


Fig. 1.7

(a) The element stiffness matrices are

$$\mathbf{k}^1 = \frac{70 \times 10^3 \times 2400}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \begin{matrix} 1 & 2 \end{matrix} \leftarrow \text{Global dof}$$

and

$$\mathbf{k}^2 = \frac{200 \times 10^3 \times 600}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \begin{matrix} 2 & 3 \end{matrix}$$

The structural stiffness matrix that is assembled from k^1 and k^2 is

$$\mathbf{K} = 10^6 \begin{bmatrix} 0.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 0.30 \end{bmatrix} \quad \begin{matrix} 1 & 2 & 3 \end{matrix}$$

The global load vector is

$$F = [0, 200 \text{ x } 10^3, 0]^T$$

Now dofs 1 and 3 are fixed. When using the penalty approach, therefore, a large numberC is added to thefirst and third diagonal elements of K. Choosing C

$$C = [0.86 \times 10^6] \times 10^4$$

Thus, the modified stiffness matrix is

$$\mathbf{K} = 10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix}$$

The finite element equations are given by

$$10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{Bmatrix}$$

which yields the solution

$$Q = [15.1432 \times 10^{-6}, 0.23257, 8.1127 \times 10^{-6}]\text{mm}$$

(b) The element stresses are

$$\sigma_1 = 70 \times 10^3 \times \frac{1}{300}[-1 \quad 1]\begin{Bmatrix} 15.1432 \times 10^{-6} \\ 0.23257 \end{Bmatrix}$$

$$= 54.27 \text{ MPa}$$

where 1 MPa = 10^6 N/m^2= 1N/mm^2. Also,

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{400}[-1 \quad 1]\begin{Bmatrix} 0.23257 \\ 8.1127 \times 10^{-6} \end{Bmatrix}$$

$$= -116.29 \text{ MPa}$$

(c) The reaction forces are

$$R_1 = - CQ_1$$

$$= -[0.86 \times 10^{10}] \times 15.1432 \times 10^6$$

$$= - 130.23 \times 10^3 \text{ N}$$

$$R_3 = - CQ_3$$

$$= -[0.86 \times 10^{10}] \times 8.1127 \times 10^6$$

$$= - 69.77 \times 10^3 \text{ N}$$

Example 1.4:In Fig. (a), a load P = 60 x 10^3 N is applied as shown. Determine the displacement field,stress and support reactions in the body. Take E = 20 x 10^3N/mm^2.

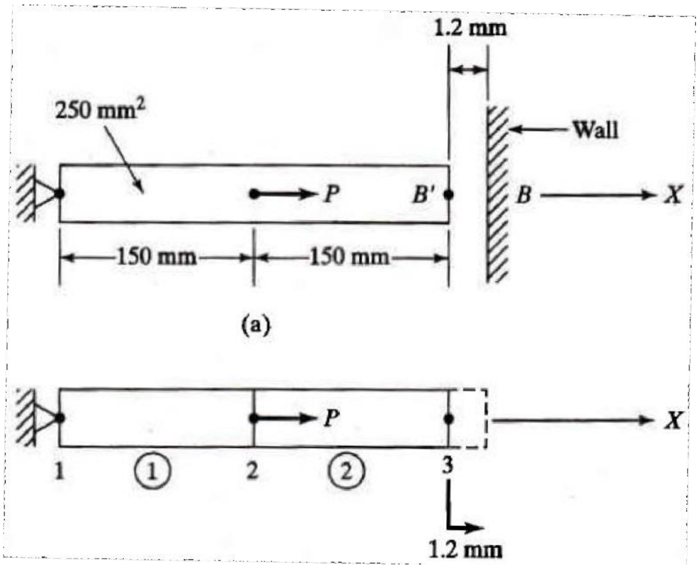


Fig.

The boundary conditions are $Q_1 = 0$ and $Q_3 = 1.2$ mm. The structural stiffness matrix K is

$$\mathbf{K} = \frac{20 \times 10^3 \times 250}{150} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

andthe global load vector F is

$$F = [0, 60 \times 10^3, 0]^T$$

In the penalty approach, the boundary conditions $Q_1 = 0$ and $Q_3 = 1.2$imply the followingmodifications: A large number C chosen here as $C = (2/3) \times 10^{10}$, is added on to the 1st and 3rd diagonal elements of K. Also, the number (C x 1.2) gets added on to the 3rd component of F. Thus, the modified equations are

The solution is

$$\frac{10^5}{3} \begin{bmatrix} 20001 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 20001 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60.0 \times 10^3 \\ 80.0 \times 10^7 \end{Bmatrix}$$

$$Q = [7.49985 \times 10^{-5}, 1.500045, 1.200015]^T \text{mm}$$

The element stresses are

$$\sigma_1 = 200 \times 10^3 \times \frac{1}{150} [-1 \quad 1] \begin{Bmatrix} 7.49985 \times 10^{-5} \\ 1.500045 \end{Bmatrix}$$

$$= 199.996 \text{ MPa}$$

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{150} [-1 \quad 1] \begin{Bmatrix} 1.500045 \\ 1.200015 \end{Bmatrix}$$

$$= -40.004 \text{ MPa}$$

The reaction forces are

$$R_1 = -C \times 7.49985 \times 10^{-5}$$

$$= -49.999 \times 10^3 \text{N}$$

$$R_3 = -C \times (1.200015 - 1.2)$$

$= -10.001 \times 10^3 N$

Effect of Temperature on Elements:

When any material is subjected to a thermal stress, the thermal load is additional load acting on every element. This load can be calculated by using thermal expansion of the material due to the rise in. temperature.

Thermal stress in material can be given by

$\zeta_t = E\varepsilon_t$

Where ε_t= thermal strain

E = modulus of elasticity

$\varepsilon_t = \alpha \Delta t$

α= coefficient of linear expansion of material

Δt= change in temperature of material.

Then the thermal load is given by

Where, $Ft = \zeta_t A = AE\alpha \Delta t$

A = Area of the bar.

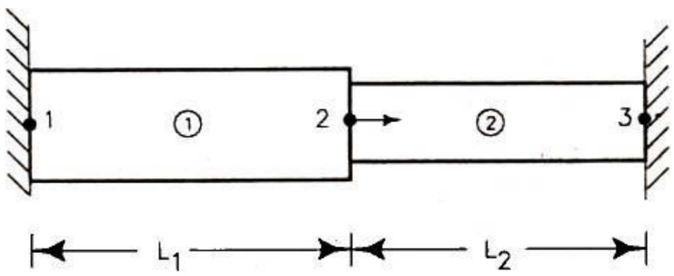


Fig.

Consider the horizontal step bar supported at two ends is subjected to athermal stress and load P at node 2 as shown in Fig. .

Thermal load in element 1

$$[F_1] = \begin{bmatrix} Ft_1 \\ Ft_{12} \\ 0 \end{bmatrix} = A_1 \; E \; \alpha \; \Delta t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Thermal load in element 2

$$[F_2] = \begin{bmatrix} 0 \\ Ft_{21} \\ Ft_3 \end{bmatrix} = A_2 \; E \; \alpha \; \Delta t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[F] = [F_1] + [F_2] + \begin{bmatrix} 0 \\ P \\ 0 \end{bmatrix} = \begin{bmatrix} -A_1 E\alpha\Delta t \\ A_1 E\alpha\Delta t - A_2 E\alpha\Delta t + P \\ A_2 E\alpha\Delta t \end{bmatrix}$$

Example 1.5 : An axial load $P = 300 \times 10^3$ N is applied at 20°C to the rod as shown in Fig. .The temperature is then raised to 60°C.

(a) Assemble the K and F matrices.

(b) Determine the nodal displacements and element stresses.

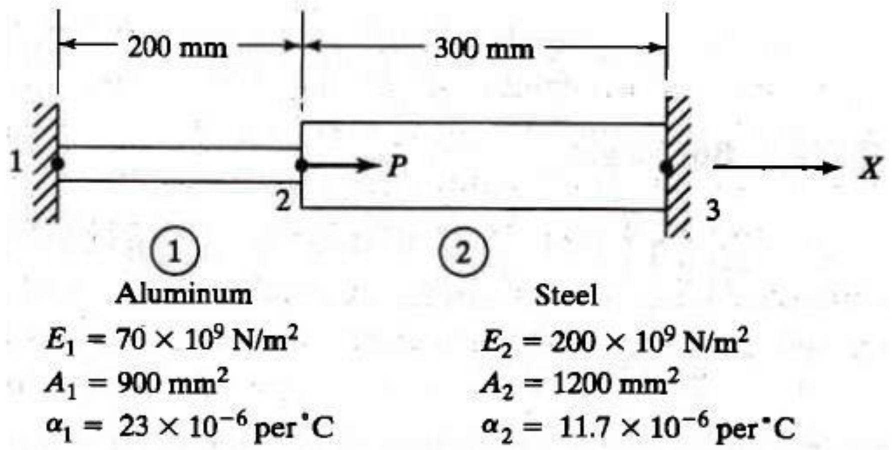


Fig.

(a) The element stiffness matrices are

$$\mathbf{k}^1 = \frac{70 \times 10^3 \times 900}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{N/mm}$$

$$\mathbf{k}^2 = \frac{200 \times 10^3 \times 1200}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{N/mm}$$

$$\mathbf{K} = 10^3\begin{bmatrix} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{bmatrix} \text{N/mm}$$

Now, in assembling F, both temperature and point load effects have to be considered.

The element temperature forces due to ΔT = 40°C are obtained as

$$\boldsymbol{\Theta}^1 = 70 \times 10^3 \times 900 \times 23 \times 10^{-6} \times 40\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}\begin{matrix} 1 \\ 2 \end{matrix} \text{ N} \quad \leftarrow \text{Global dof}$$

$$\boldsymbol{\Theta}^2 = 200 \times 10^3 \times 1200 \times 11.7 \times 10^{-6} \times 40\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}\begin{matrix} 2 \\ 3 \end{matrix} \text{ N}$$

Upon assembling $\boldsymbol{\Theta}^1$, $\boldsymbol{\Theta}^2$, and the point load, we get

$$\mathbf{F} = 10^3\begin{Bmatrix} -57.96 \\ 57.96 - 112.32 + 300 \\ 112.32 \end{Bmatrix}$$

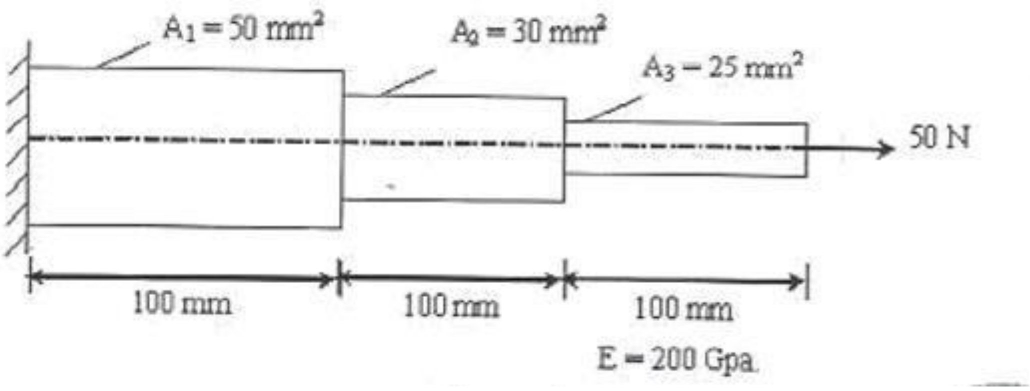
$$F = 10^3[-57.96,\ 245.64,\ 112.32]^T \text{N}$$

(b) The elimination approach will now be used to solve for the displacements. Since dofs 1 and 3 are fixed, the first and third rows and columns of K, together with the first and third components of F, are deleted. This results in the scalar equation

UNIT I
TUTORIAL QUESTIONS

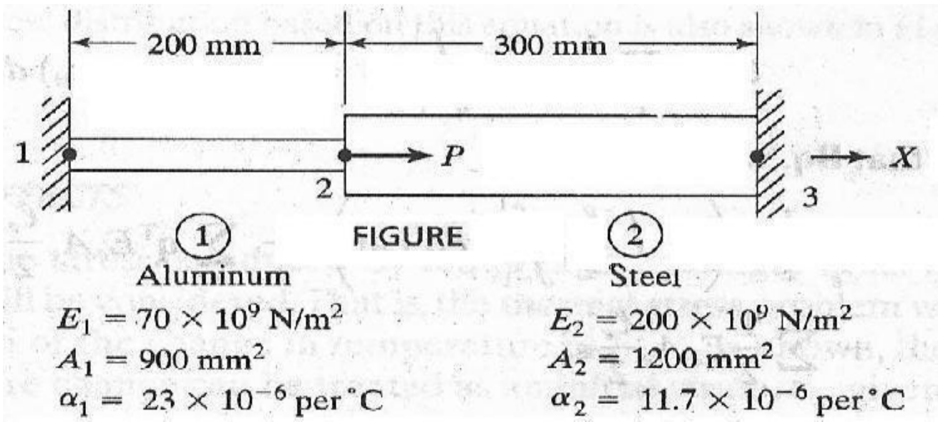
1. Derive the equations of equilibrium for 3D body
2. Explain about plane stress and plane strain
3. Describe advantages, disadvantages and applications of finite element analysis
4. The following equation is available for a physical phenomena
   $\frac{d^2 y}{dx^2}$ -10 =5; $0 \leq x\, x^2 < 1$, Boundary Conditions; y(0) =0, y(1) =0, Using Galarkin method of weighted residual find an approximate solution of the above differentialequation

5. Use Finite Element method Calculate nodal displacements and element stresses



UNIT I
ASSIGNMENT QUESTIONS

1. Describe the standard procedure to be followed for understanding the finiteelement method step by step with suitable example.
2. Derive the stiffness matrix of axial bar element with quadratic shape functions based on firstprinciples.
3. An axial load P=300KN is applied at 20^0 C to the rod as shown in Figure below.The temperature is the raised to 60^0 C .
   a) Assemble the K and F matrices.
   b) Determine the nodal displacements and stresses.



UNIT 2

TRUSSES
&
BEAMS

Syllabus

TRUSSES: Element matrices, assembling of global stiffness matrix, solution fordisplacements, reaction, stresses.
BEAMS: Element matrices, assembling of global stiffness matrix, solution for displacements, reaction, stresses.

OBJECTIVE:

To learn the application of FEM equations for trusses and Beams

OUTCOME:

Derive element matrices to find stresses in trusses and Beams

UNIT II

Analysis of Trusses

- The links of a truss are two-force members, where the direction of loading is along the axis of the member. Every truss element is in direct tension or compression.
- All loads and reactions are applied only at the joints and all members are connected together at their ends by frictionless pin joints. This makes the truss members very similar to a 1D spar element.

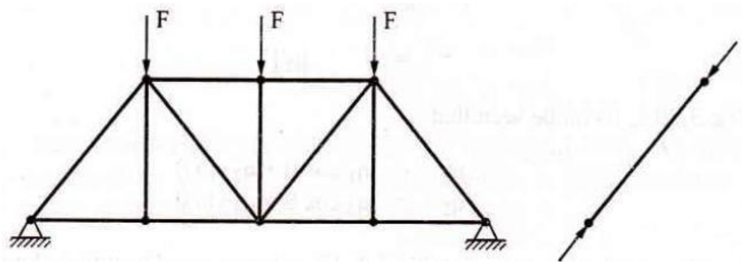


Fig. 1.28 Truss

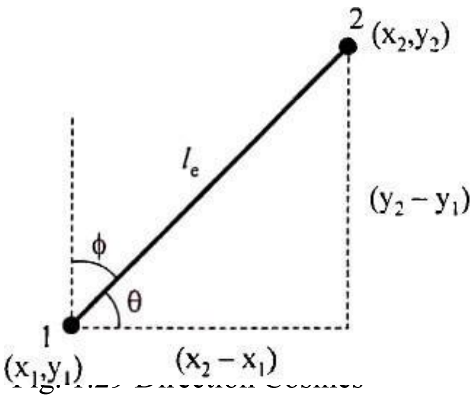


Fig. 1.29 Direction Cosines

The direction cosines l and m can be expressed as:

$$l = \cos\theta = \frac{x_2 - x_1}{l_e}$$

$$m = \cos\phi = \sin\theta = \frac{y_2 - y_1}{l_e}$$

$$\sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$$

$q_1'' = q_1 l + q_2 m$

$q_2'' = q_3 l + q_4 m$

$$[K] = \frac{A_e E_e}{L_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$\sigma = \frac{E_e}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} q$$

- **Thermal Effect In Truss Member**

(1) Thermal Load , P = $A_e E_e \varepsilon_e \begin{bmatrix} -l \\ -m \\ l \\ m \end{bmatrix}$

(2) Stress for an element, $\sigma = \frac{E_e}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} q - E_e \alpha \Delta t$

(3) Remaining steps will be same as earlier.

Example 1.6: A two member truss is as shown in Fig. The cross-sectional area ofeach member is 200 mm^2 and the modulus of elasticity is 200 GPa. Determine the deflections,reactions and stresses in each of the members.

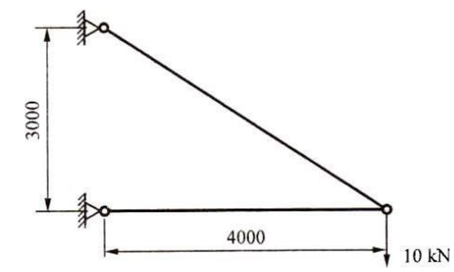


Fig.

In globalterms, each node would have 2 dof. These dof are marked as shown in Fig.1.31. The position of the nodes, with respect to origin (considered at node l) are as tabulated below:

Node	Xi	Yi
1	0	0
2	4000	0
3	0	3000

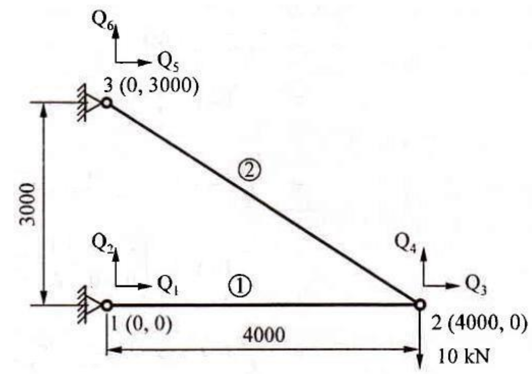


Fig.

For all elements, A=200 mm^2

and E= 200 x 10^3 N/mm^2

The element connectivity table with the relevant terms are:

Element	N_i	N_j	$l_e = \sqrt{(x_j - x_i)^2 - (y_j - y_i)^2}$	$\frac{A_e E_e}{l_e}$	$l = \frac{x_j - x_i}{l_e}$	$m = \frac{y_j - y_i}{l_e}$	l^2	m^2	lm
(1)	1	2	$\sqrt{(4000 - 0)^2 - (0 - 0)^2}$ $= 4000$	10000	$\frac{4000 - 0}{4000} = 1$	$\frac{0 - 0}{4000} = 0$	1	0	0
(2)	2	3	$\sqrt{(0 - 4000)^2 + (3000 - 0)^2}$ $= 5000$	8000	$-\frac{4000}{5000} = -0.8$	$\frac{3000}{5000} = 0.6$	0.64	0.36	-0.48

As each node has two dof in global form, for every element, the element stiffness matrix would be in a 4 x 4 form. For element 1 defined by nodes l-2,the dof are Q_1, Q_2, Q_3 andQ_4 and that for element 2 defined by nodes 2-3, would be Q_3, Q_4, Q_5 andQ_6.

Element 1: The element stiffness matrix would be :

$$\begin{array}{c} \text{Node1} \quad \text{Node2} \\ \overbrace{1 \quad 2} \quad \overbrace{3 \quad 4} \Leftarrow \text{Global dof} \end{array}$$

$$K^1 = 10 \times 10^3 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \Downarrow \\ 4 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 3 & 4 & \Leftarrow \text{Global dof} \end{matrix}$$

$$= 10^3 \begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \Downarrow \\ 4 \end{matrix}$$

Element 2: The element stiffness matrix would be :

$$\begin{array}{c} \text{Node2} \quad \quad \text{Node3} \\ \overbrace{3 \quad \quad 4} \quad \overbrace{5 \quad \quad 6} \Leftarrow \text{Global dof} \end{array}$$

$$K^2 = 8 \times 10^3 \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \Downarrow \\ 6 \end{matrix}$$

$$\begin{matrix} 3 & 4 & 5 & 6 & \Leftarrow \text{Global dof} \end{matrix}$$

$$= 10^3 \begin{bmatrix} 5.12 & -3.84 & -5.12 & 3.84 \\ -0.48 & 2.88 & 3.84 & -2.88 \\ -5.12 & 3.84 & 5.12 & -3.84 \\ 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \Downarrow \\ 6 \end{matrix}$$

The global stiffness matrix would be :

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & \Leftarrow \text{Global dof} \end{matrix}$$

$$K = 10^3 \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & (10+5.12) & (0-3.84) & -5.12 & 3.84 \\ 0 & 0 & (0-3.84) & (0+2.88) & 3.84 & -2.88 \\ 0 & 0 & -5.12 & 3.84 & 5.12 & -3.84 \\ 0 & 0 & 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \Downarrow \\ 5 \\ 6 \end{matrix}$$

$$
\begin{array}{c} \\ = 10^3 \end{array}
\begin{array}{c}
\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \Leftarrow \text{Global dof} \\
\begin{bmatrix}
10 & 0 & -10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-10 & 0 & 15.12 & -3.84 & -5.12 & 3.84 \\
0 & 0 & -3.84 & 2.88 & 3.84 & -2.88 \\
0 & 0 & -5.12 & 3.84 & 5.12 & -3.84 \\
0 & 0 & 3.84 & -2.88 & -3.84 & 2.88
\end{bmatrix}
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \Downarrow
\end{array}
$$

In this case, node 1 and node 3 are completely fixed and hence,

$$Q_1 = Q_2 = Q_5 = Q_6 = 0$$

Hence, rows and columns 1,2,5 and 6 can be eliminated

Also the external nodal forces,

$F_1 = F_2 = F_3 = F_5 = F_6 = 0$

$F4 = -10 \times 10^3$ N

The global force vector would be,

$$
\mathbf{F} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -10\times10^3 \\ 0 \\ 0 \end{Bmatrix}
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \Downarrow \quad \text{Global dof}
$$

In global form, after using the elimination approach

$$\mathbf{KQ} = \mathbf{F}$$

$$10^3 \begin{bmatrix} 15.12 & -3.84 \\ -3.84 & 2.88 \end{bmatrix} \begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10\times10^3 \end{Bmatrix}$$

$$10^3 (15.12\, Q_3 - 3.84\, Q_4) = 0$$

$$Q_3 = 0.254\, Q_4$$

$10^3 (-3.84\, Q_3 + 2.88\, Q_4) = -10 \times 10^3$

$-3.84\, Q_3 + 2.88\, Q_4 = -10$

$-3.84\, (0.254\, Q_4) + 2.88\, Q_4 = -10$

$Q_4 = -5.25$mm

$Q_3 = -1.334$mm

The reactions can be found by using the equation:

$R = KQ - F$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_5 \\ R_6 \end{Bmatrix} = 10^3 \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5.12 & 3.84 & 5.12 & -3.84 \\ 0 & 0 & 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1.334 \\ -5.25 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$R_1 = -10 \times 10^3 \times (-1.334) = 13340$ N

$R_2 = 0$ N

$R_5 = -5.12 \times 10^3 \times (-1.334) + 3.84 \times 10^3 \times (-5.25) = -13340$ N

$R_6 = 3.84 \times 10^3 \times (-1.334) - 2.88 \times 10^3 \times (-5.25) = 9997.44$ N

To determine stresses: $\sigma = \frac{E_e}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} q$

Element 1:

$$\sigma_1 = \frac{200 \times 10^3}{4000} [-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} 0 \\ 0 \\ -1.334 \\ -5.25 \end{Bmatrix}$$

$$= -66.7 \text{ N/mm}^2$$

Element 2:

$$\sigma_2 = \frac{200 \times 10^3}{5000} [0.8 \quad -0.6 \quad -0.8 \quad 0.6] \begin{Bmatrix} -1.334 \\ -5.25 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 83.312 \text{ N/mm}^2$$

Example :Consider the four-bar truss shown in Fig. It is given that $E = 29.5 \times 10^6$ psi and A_e = lin.^{2} for all elements. Complete the following:

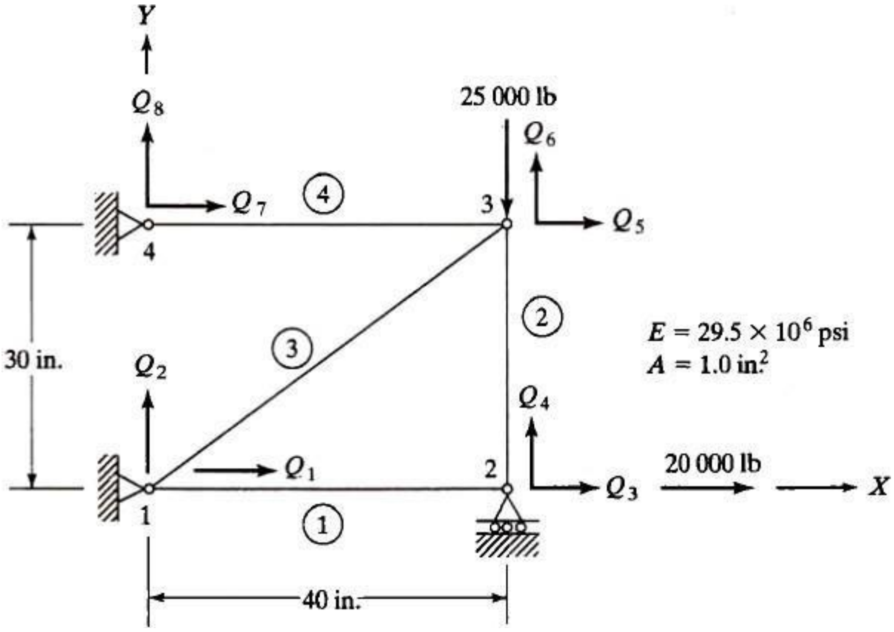
(a) Determine the element stiffness matrix for each element.

(b) Assemble the structural stiffness matrix K for the entire truss

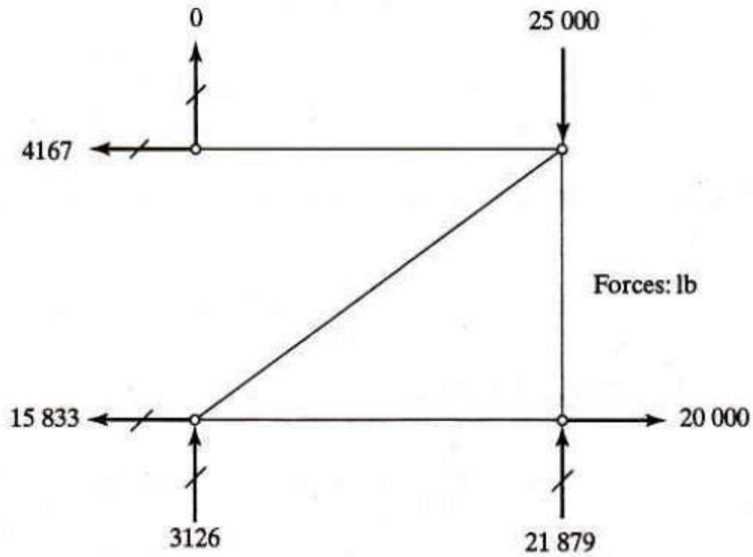
(c) Using the elimination approach, solve for the nodal displacement.

(d) Recover the stresses in each element.

(e) Calculate the reaction forces.



(a)



(b)

Fig

(a) It is recommended that a tabular form be used for representing nodal coordinate data and element information. The nodal coordinate data are as follows:

Node	x	y
1	0	0
2	40	0
3	40	30
4	0	30

The element connectivity table is

Element	1	2
1	1	2
2	3	2
3	1	3
4	4	3

Note that the user has a choice in defining element connectivity. For example, the connectivityof element 2 can be defined as 2-3 instead of 3-2 as in the previous table. However,calculations of the direction cosines will be consistent with the adopted connectivityscheme. Using formulas,together with the nodal coordinate data andthe given element connectivity information, we obtain the direction cosines table:

Element	l_e	l	m
1	40	1	0
2	30	0	-1
3	50	0.8	0.6
4	40	1	0

For example, the direction cosines of elements 3 are obtained as

$l = (x_3 - x_1)l_e = (40 - 0)/50 = 0.8$ and $m = (y_3 - y_1)l_e = (30 - 0)/50 = 0.6$.

Now, the element stiffness matrices for element 1 can be written as

$$\begin{matrix} & 1 & 2 & 3 & 4 & \leftarrow \text{Global dof} \end{matrix}$$

$$\mathbf{k}^1 = \frac{29.5 \times 10^6}{40}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

The global dofs associated with element 1, which is connected between nodes 1 and2,are indicated in k^1 earlier.These global dofs are shown in Fig. 1.32(a) and assistin assembling the various element stiffness matrices.The element stiffness matrices of elements 2,3and 4 are as follows:

$$\begin{matrix} 5 & 6 & 3 & 4 \end{matrix}$$

$$\mathbf{k}^2 = \frac{29.5 \times 10^6}{30}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{matrix} 5 \\ 6 \\ 3 \\ 4 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 5 & 6 \end{matrix}$$

$$\mathbf{k}^3 = \frac{29.5 \times 10^6}{50}\begin{bmatrix} .64 & .48 & -.64 & -.48 \\ .48 & .36 & -.48 & -.36 \\ -.64 & -.48 & .64 & .48 \\ -.48 & -.36 & .48 & .36 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}$$

$$\mathbf{k}^4 = \frac{29.5 \times 10^6}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 7 \\ 8 \\ 5 \\ 6 \end{matrix}$$

(columns: 7, 8, 5, 6)

(b) The structural stiffness matrix K is now assembled from the element stiffness matrices.By adding the element stiffness contributions, noting the element connectivity, we get

$$\mathbf{K} = \frac{29.5 \times 10^6}{600} \begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ -15.0 & 0 & 15.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ -7.68 & -5.76 & 0 & 0 & 22.68 & 5.76 & -15.0 & 0 \\ -5.76 & -4.32 & 0 & -20.0 & 5.76 & 24.32 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$

(columns: 1, 2, 3, 4, 5, 6, 7, 8)

(c) The structural stiffness matrix K given above needs to be modified to account for the boundary conditions. The elimination approach will be used here. The rows and columns corresponding to dofs 1, 2, 4, 7, and 8, which correspond to fixed supports, are deleted from the K matrix. The reduced finite element equations are given as

$$\frac{29.5 \times 10^6}{600} \begin{bmatrix} 15 & 0 & 0 \\ 0 & 22.68 & 5.76 \\ 0 & 5.76 & 24.32 \end{bmatrix} \begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 20\,000 \\ 0 \\ -25\,000 \end{Bmatrix}$$

Solution of these equations yields the displacements

$$\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 27.12 \times 10^{-3} \\ 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \end{Bmatrix} \text{in.}$$

The nodal displacement vector for the entire structure can therefore be written as

$Q = [0, 0, 27.12 \text{x } 10^{-3}, 0, 5.65 \text{ x } 10^{-3}, -22.25 \text{ x } 10^{-3}, 0, 0]^T$ in.

(d) The stress in each element can now be determined as shown below.
The connectivity of element 1 is 1 - 2. Consequently, the nodaldisplacementvectorforelementlisgivenby $q = [0, 0, 27.72 \text{ x } 10^{-3}, 0]^T$

$$\sigma_1 = \frac{29.5 \times 10^6}{40} [-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} 0 \\ 0 \\ 27.12 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$= 20\,000.0 \text{ psi}$$

The stress in member 2 is given by

$$\sigma_2 = \frac{29.5 \times 10^6}{30}[0 \quad 1 \quad 0 \quad -1]\begin{Bmatrix} 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \\ -27.12 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$= -21\,880.0 \text{ psi}$$

Following similar steps, we get

ζ_3 = 5208.0Psi

ζ_4= 4L67.0Psi

(e) The final step is to determine the support reactions. We need to determine the reaction forces along dofs 1, 2, 4, 7and 8, which correspond to fixed supports. These are obtained by substituting for Q into the original finite element equation R = KQ - F. In this substitution, only those rows of K corresponding to the support dofs are needed, and F = 0 for those dofs. Thus, we have

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \frac{29.5 \times 10^6}{600}\begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 27.12 \times 10^{-3} \\ 0 \\ 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

Which results in

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} -15833.0 \\ 3126.0 \\ 21879.0 \\ -4167.0 \\ 0 \end{Bmatrix} \text{lb}$$

UNIT II

BEAMS

Derivation of Shape Function for Beam Element [Fourth Order Beam Equation]

Consider the beam element as shown in Fig. The beam is of length L with axial local co-ordinate x and transverse local co-ordinate y. The local transverse nodal displacements are given by d_{1y} and d_{2y}. The rotations are given by ϕ_1 and ϕ_2. The local nodal forces are given by F_{1y} and F_{2y}. The bending moments are given by m_1 and m_2.

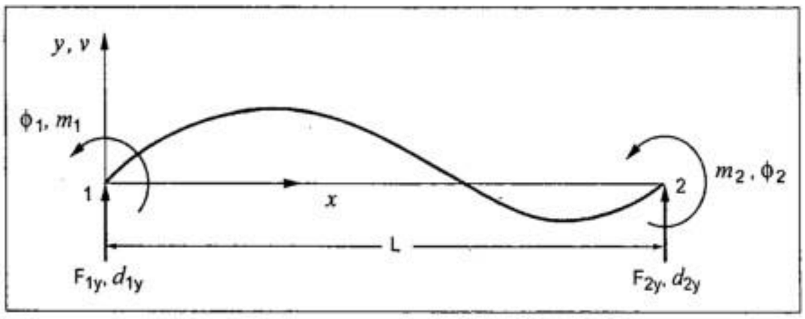


Fig. Beam element with positive nodal displacements, rotations, forces, and moments

At all nodes, the following sign conversions are used.

(*i*) Moments are positive in the counterclockwise direction.

(*ii*) Rotations are positive in the counterclockwise direction.

(*iii*) Forces are positive in the positive y direction.

(*iv*) Displacements are positive in the positive y direction.

Fig. indicates the sign conventions used in simple beam theory for positive shear forces F and bending moments m.

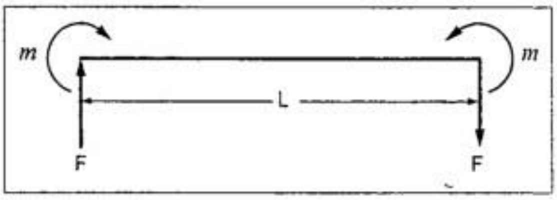


Fig. Beam theory sign conventions for shear forces and bending moments

Assume the transverse displacement variation through the element length to be

$$v(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4$$

We express v in terms of the nodal degrees of freedom d_{1y}, d_{2y}, ϕ_1 and ϕ_2 as follows:

At $x = 0$,

$$v(0) = a_4 = d_{1y}$$

$$\frac{dv_{(x)}}{dx} = 3 a_1 x^2 + 2 a_2 x + a_3$$

$$\frac{dv_{(0)}}{dx} = a_3 = \phi_1$$

When $x = L$,

$$v(L) = a_1 L^3 + a_2 L^2 + a_3 L + a_4 = d_{2y}$$

$$\frac{dv_{(L)}}{dx} = 3a_1 L^2 + 2a_2 L + a_3 = \phi_2$$

where $\phi = \frac{dv}{dx}$

Finding a_1 and a_2 in terms of d_{1y}, d_{2y}, ϕ_1 and ϕ_2 by using the above equations

$$\Rightarrow \quad d_{2y} = a_1 L^3 + a_2 L^2 + a_3 L + a_4$$

$$= a_1 L^3 + a_2 L^2 + a_3 L + d_{1y} \qquad [\because a_4 = d_{1y}]$$

$$\Rightarrow \quad (d_{2y} - d_{1y}) = a_1 L^3 + a_2 L^2 + \phi_1 L$$

$$\Rightarrow \quad (d_{2y} - d_{1y} - \phi_1 L) = a_1 L^3 + a_2 L^2$$

$$\Rightarrow \quad \frac{1}{L}(d_{2y} - d_{1y} - \phi_1 L) = a_1 L^2 + a_2 L$$

$$\Rightarrow \quad \phi_2 = 3a_1 L^2 + 2a_2 L + a_3$$

$$= 3a_1 L^2 + 2a_2 L + \phi_1 \qquad [\because a_3 = \phi_1]$$

$$\Rightarrow \quad \phi_2 - \phi_1 = 3a_1 L^2 + 2a_2 L$$

Equation

$$\Rightarrow \quad \frac{3}{L}(d_{2y} - d_{1y} - \phi_1 L) = 3a_1 L^2 + 3a_2 L$$

Solving equation

$$\phi_2 - \phi_1 = 3a_1 L^2 + 2a_2 L$$

$$\frac{3}{L}(d_{2y} - d_{1y} - \phi_1 L) = 3a_1 L^2 + 3a_2 L$$

Subtracting,

$$\phi_2 - \phi_1 - \frac{3}{L}(d_{2y} - d_{1y} - \phi_1 L) = -a_2 L$$

$$\phi_2 - \phi_1 - \frac{3}{L}(d_{2y} - d_{1y}) + \frac{3}{L}\phi_1 L = -a_2 L$$

$$\phi_2 - \phi_1 - \frac{3}{L}(d_{2y} - d_{1y}) + 3\phi_1 = -a_2 L$$

$$\phi_2 + 2\phi_1 - \frac{3}{L}(d_{2y} - d_{1y}) = -a_2 L$$

$$\frac{1}{L}(\phi_2 + 2\phi_1) - \frac{3}{L^2}(d_{2y} - d_{1y}) = -a_2$$

$$\Rightarrow \quad \frac{-1}{L}(\phi_2 + 2\phi_1) + \frac{3}{L^2}(d_{2y} - d_{1y}) = a_2$$

$$\Rightarrow \quad \frac{-1}{L}(\phi_2 + 2\phi_1) - \frac{3}{L^2}(d_{1y} - d_{2y}) = a_2$$

$$\Rightarrow \quad a_2 = \frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)$$

Substitute a_2 value in equation

$$\Rightarrow \quad \phi_2 = 3a_1 L^2 + 2L\left[\frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)\right] + a_3$$

$$= 3a_1 L^2 - \frac{6}{L}(d_{1y} - d_{2y}) - 2(2\phi_1 + \phi_2) + \phi_1 \qquad [\because a_3 = \phi_1]$$

$$\Rightarrow \quad \phi_2 - \phi_1 = 3a_1 L^2 - \frac{6}{L}(d_{1y} - d_{2y}) - 4\phi_1 - 2\phi_2$$

$$\Rightarrow \quad 3\phi_1 + 3\phi_2 = 3a_1 L^2 - \frac{6}{L}(d_{1y} - d_{2y})$$

$$\Rightarrow \quad 3a_1 L^2 = 3\phi_1 + 3\phi_2 + \frac{6}{L}(d_{1y} - d_{2y})$$

$$a_1 L^2 = \phi_1 + \phi_2 + \frac{2}{L}(d_{1y} - d_{2y})$$

$$\Rightarrow \quad a_1 = \frac{1}{L^2}(\phi_1 + \phi_2) + \frac{2}{L^3}(d_{1y} - d_{2y})$$

$$\Rightarrow \quad \boxed{a_1 = \frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)}$$

Substitute a_1, a_2, a_3 and a_4 values in equation

$$v(x) = \left[\frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[\frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + d_{1y}$$

$$[\because a_3 = \phi_1;\ a_4 = d_{1y}]$$

In matrix form, $\quad v(x) = [N]\{d\}$

$$\Rightarrow \quad v(x) = [N_1 \ N_2 \ N_3 \ N_4]\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

$$\Rightarrow \quad v(x) = N_1 d_{1y} + N_2 \phi_1 + N_3 d_{2y} + N_4 \phi_2$$

where N_1, N_2, N_3 and N_4 are shape functions for beam element.

Stiffness Matrix [K] for Beam Element

The stiffness matrix for the beam element is derived by using a direct equilibrium approach and beam theory sign conversions.

We know that,

Transverse displacement

$$v(x) = \left[\frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]x^3 + \left[\frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)\right]x^2 + \phi_1 x + d_{1y}$$

$$\Rightarrow \quad \frac{dv_{(x)}}{dx} = 3x^2\left[\frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)\right] + 2x\left[\frac{-3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + \phi_2)\right] + \phi_1$$

$$+ 2\left[\frac{-3}{L^2}(d_{1y}-d_{2y})-\frac{1}{L}(2\phi_1+\phi_2)\right]$$

$$\frac{d^3v(x)}{dx^3} = 6\left[\frac{2}{L^3}(d_{1y}-d_{2y})+\frac{1}{L^2}(\phi_1+\phi_2)\right]$$

Put $x = 0$ in equation (

$$\Rightarrow \frac{d^2v(0)}{dx^2} = 0 + 2\left[\frac{-3}{L^2}(d_{1y}-d_{2y})-\frac{1}{L}(2\phi_1+\phi_2)\right]$$

$$= \frac{-6}{L^2}(d_{1y}-d_{2y})-\frac{2}{L}(2\phi_1+\phi_2)$$

$$= \frac{1}{L^3}\left[-6Ld_{1y}+6Ld_{2y}-4L^2\phi_1-2L^2\phi_2\right]$$

$$\frac{d^2v(0)}{dx^2} = \frac{1}{L^3}\left[-6Ld_{1y}-4L^2\phi_1+6Ld_{2y}-2L^2\phi_2\right]$$

Put $x = L$ in equation

$$\Rightarrow \frac{d^2v(L)}{dx^2} = 6L\left[\frac{2}{L^3}(d_{1y}-d_{2y})+\frac{1}{L^2}(\phi_1+\phi_2)\right]$$

$$+ 2\left[\frac{-3}{L^2}(d_{1y}-d_{2y})-\frac{1}{L}(2\phi_1+\phi_2)\right]$$

$$= \frac{12L}{L^3}(d_{1y}-d_{2y})+\frac{6L}{L^2}(\phi_1+\phi_2)-\frac{6}{L^2}(d_{1y}-d_{2y})-\frac{2}{L}(2\phi_1+\phi_2)$$

$$= \frac{1}{L^3}[12Ld_{1y}-12Ld_{2y}+6L^2\phi_1+6L^2\phi_2-6Ld_{1y}$$

$$+6Ld_{2y}-4L^2\phi_1-2L^2\phi_2]$$

$$\frac{d^2v(L)}{dx^2} = \frac{1}{L^3}[6Ld_{1y}+2L^2\phi_1-6Ld_{2y}+4L^2\phi_2]$$

Put $x = 0$ in equation

$$\frac{d^3v(0)}{dx^3} = 6\left[\frac{2}{L^3}(d_{1y}-d_{2y})+\frac{1}{L^2}(\phi_1+\phi_2)\right]$$

$$= \frac{1}{L^3}[12d_{1y}-12d_{2y}+6L\phi_1+6L\phi_2]$$

$$\frac{d^3v(0)}{dx^3} = \frac{1}{L^3}[12d_{1y}+6L\phi_1-12d_{2y}+6L\phi_2]$$

Put $x = L$ in equation

$$\Rightarrow \frac{d^3v(L)}{dx^3} = 6\left[\frac{2}{L^3}(d_{1y}-d_{2y})+\frac{1}{L^2}(\phi_1+\phi_2)\right]$$

$$= \frac{1}{L^3}[12d_{1y}-12d_{2y}+6L\phi_1+6L\phi_2]$$

$$\frac{d^3v(L)}{dx^3} = \frac{1}{L^3}[12d_{1y}+6L\phi_1-12d_{2y}+6L\phi_2]$$

We know that,

Nodal force, $F_{1y} = EI\dfrac{d^3v(0)}{dx^3}$

$$\Rightarrow F_{1y} = \frac{EI}{L^3}[12d_{1y}+6L\phi_1-12d_{2y}+6L\phi_2]$$

Bending moment, $m_1 = -EI \dfrac{d^2 v(0)}{dx^2}$

$$= \frac{-EI}{L^3}\left[-6L\,d_{1y} - 4L^2\phi_1 + 6L\,d_{2y} - 2L^2\phi_2\right]$$

$$m_1 = \frac{EI}{L^3}\left[6L\,d_{1y} + 4L^2\phi_1 - 6L\,d_{2y} + 2L^2\phi_2\right]$$

Nodal force, $F_{2y} = -EI \dfrac{d^3 v(L)}{dx^3}$

$$= \frac{-EI}{L^3}\left[12\,d_{1y} + 6L\,\phi_1 - 12\,d_{2y} + 6L\,\phi_2\right]$$

$$F_{2y} = \frac{EI}{L^3}\left[-12\,d_{1y} - 6L\,\phi_1 + 12\,d_{2y} - 6L\,\phi_2\right]$$

Bending moment, $m_2 = EI \dfrac{d^2 v(L)}{dx^2}$

$\Rightarrow$
$$m_2 = \frac{EI}{L^3}\left[6L\,d_{1y} + 2L^2\phi_1 - 6L\,d_{2y} + 4L^2\phi_2\right]$$

Arranging to the above equation (F_{1y}, m_1, F_{2y}, m_2) in matrix form,

$\Rightarrow$
$$\begin{Bmatrix} F_{1y} \\ m_1 \\ F_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} \ldots$$

This is a finite element equation for a beam element.

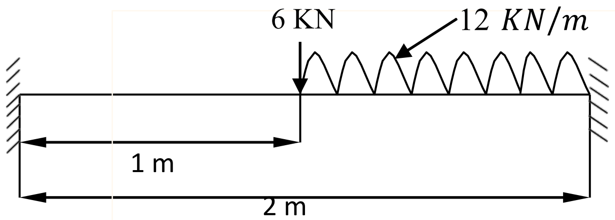
$$\text{Stiffness matrix, } [K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

where E = Young's modulus

I = Moment of inertia

L = Length of the beam

For the beam and loading shown in fig. calculate the nodal displacements.
Take [E] =210 GPa =210×10⁹ N/m^2 , [I] = 6×10⁻⁶ m⁴ NOV / DEC 2013



Given data

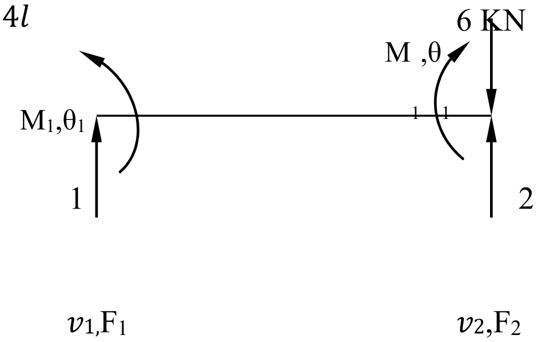
Young's modulus [E] =210 GPa =210×10^9 N/m^2
Moment of inertia [I] = 6×10^{-6} m^4
Length $[L]_1$ = 1m
Length $[L]_2$ = 1m
W=12 kN/m =12×10^3 N/m
F = 6KN

To find

- Deflection

Formula used

$$f(x)\begin{Bmatrix} \frac{-l}{2} \\ \frac{-l^2}{12} \\ \frac{-l}{2} \\ \frac{l^2}{12} \end{Bmatrix} + \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$



Solution

For element 1

$$f(x)\begin{Bmatrix} \frac{-l}{2} \\ \frac{-l^2}{12} \\ \frac{-l}{2} \\ \frac{l^2}{12} \end{Bmatrix} + \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix}$$

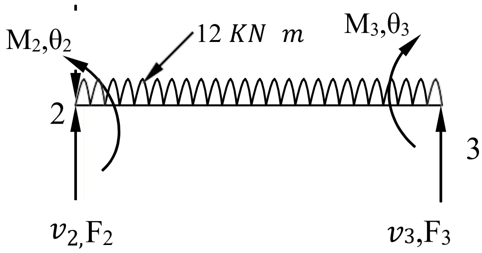
Applying boundary conditions

F_1=0N ; F_2=-6KN=-6×10^3 N; f(x)=0

$M_1=M_2=0$; $u_1=0$; $\theta_1=0$; $u_2\neq0$; $\theta_2\neq0$

$$10^3\times\begin{bmatrix}0\\0\\-6\\0\end{bmatrix}=\frac{210\times10^9\times6\times10^{-6}}{1^3}\begin{bmatrix}12 & 6 & -12 & 6\\6 & 4 & -6 & 2\\-12 & -6 & 12 & -6\\6 & 2 & -6 & 4\end{bmatrix}\begin{bmatrix}u_1\\\theta_1\\u_2\\\theta_2\end{bmatrix}$$

$$=1.26\times10^{6}\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix}\begin{Bmatrix}0\\ 0\\ u_2\\ 0\end{Bmatrix}$$



For element 2

$$f(x)\begin{Bmatrix}-\frac{l}{2}\\ -\frac{l^2}{12}\\ -\frac{l}{2}\\ \frac{l^2}{12}\end{Bmatrix}+\begin{Bmatrix}F_2\\ M_2\\ F_3\\ M_3\end{Bmatrix}=\frac{EI}{l^3}\begin{bmatrix}12 & 6l & -12 & 6l\\ 6l & 4l^2 & -6l & 2l^2\\ -12 & -6l & 12 & -6l\\ 6l & 2l^2 & -6l & 4l^2\end{bmatrix}\begin{Bmatrix}u_2\\ \theta_2\\ u_3\\ \theta_3\end{Bmatrix}$$

Applying boundary conditions

$f(x) = -12\ kN/m = 12\times10^3\ N/m$; $F_2=F_3=0=M_2=M_3$

$u_2\neq0$; $\theta_2\neq0$; $u_3=\theta_3=0$

$$10^3\times\begin{Bmatrix}-6\\ 1\\ -6\\ 1\end{Bmatrix}+\begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}=1.26\times10^{6}\times\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & 6 & 12 & -6\\ 6 & 4 & -6 & 4\end{bmatrix}\begin{Bmatrix}u_2\\ \theta_2\\ 0\\ 0\end{Bmatrix}$$

$$10^3\times\begin{Bmatrix}-6\\ -1\\ -6\\ 1\end{Bmatrix}=1.26\times10^{6}\times\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & 6 & 12 & -6\\ 6 & 4 & -6 & 4\end{bmatrix}\begin{Bmatrix}u_2\\ \theta_2\\ 0\\ 0\end{Bmatrix}$$

Assembling global matrix

$$10^3\times\begin{Bmatrix}0\\ 0\\ -12\\ -1\\ -6\\ 1\end{Bmatrix}=1.26\times10^{6}\times\begin{bmatrix}12 & 6 & -12 & -6 & 0 & 0\\ 6 & 4 & -6 & 2 & 0 & 0\\ -12 & -6 & 24 & 0 & -12 & 6\\ 6 & 2 & 0 & 8 & -6 & 2\\ 0 & 0 & -12 & -6 & 12 & -6\\ 0 & 0 & 6 & 2 & -6 & 4\end{bmatrix}\begin{Bmatrix}0\\ 0\\ u_2\\ \theta_2\\ 0\\ 0\end{Bmatrix}$$

Solving matrix

$-12\times10^3=1.26\times10^6\times24u_2=0$; $u_2=-3.96\times10^{-4}$m

$-1\times10^3=1.26\times10^6\times8\theta_2=0$; $\theta_2=-9.92$rad

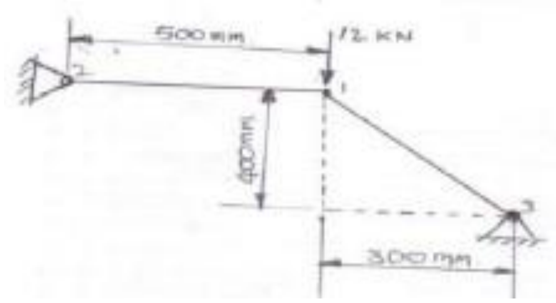
Result

$\theta_2=-9.92$rad

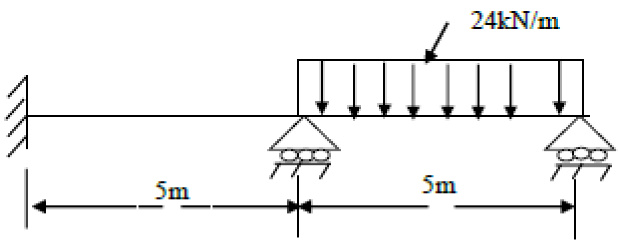
$u_2=-3.96\times10^{-4}$m

UNIT II
Tutorial Questions

1. For the two bar truss shown in figure, determine the displacement at node 1 and stresses in element2, Take E=70GPa, A= 200mm^2.



2. For the beam loaded as shown in figure, determine the slope at the simple supports. Take E=200GPa, I=4x10^{6}m^4.



3. For a beam and loading shown in fig., determine the slopes at 2 and 3 and the vertical deflection at the midpoint of the distributed load.

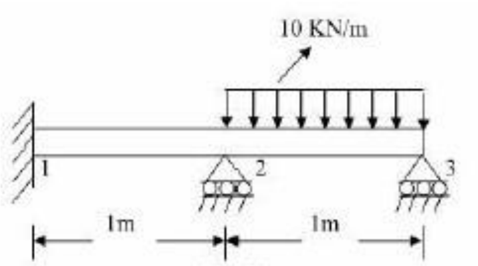
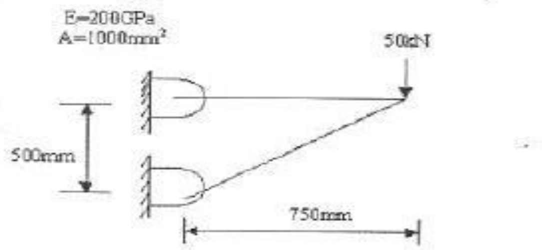


Fig.5

4.

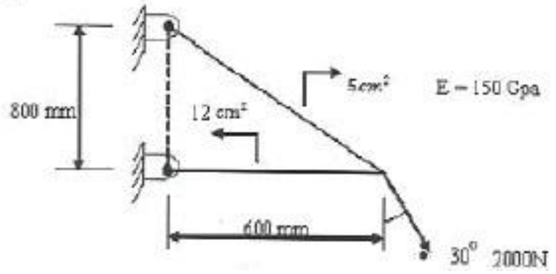
Determine the stiffness matrix, stresses and reactions in the truss structure shown in figure



UNIT II
Assignment Questions

1. Derive the stiffness matrix for Truss element
2. Derive the stiffness matrix for Beam element
3.

Calculate the nodal displacement, stresses and support reactions for the truss shown in Figure.



UNIT 3

TWO DIMENSIONL PROBLEMS
&
AXI-SYMMETRIC MODELS

Syllabus

Two Dimensional Problems: Basic concepts of plane stress and plane strain, stiffness matrix of CST element, finite element solution of plane stress problems. Axi-Symmetric Model: Finite element modelling of axi-symmetric solids subjected to axi- symmetric loading with triangular elements.

OBJECTIVE:

To learn the applications of FEM equations in 2D Plane problems with CST elements.

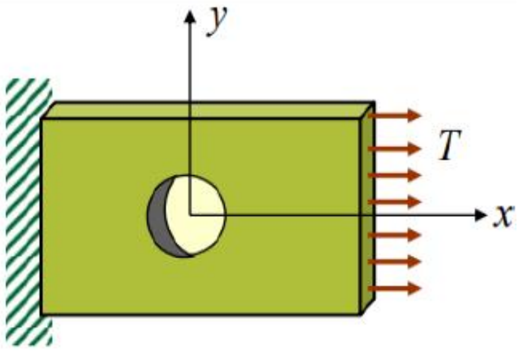
OUTCOME:

Formulate FE characteristic equations for axisymmetric problems and analyze plain stress, plain strain and Derive element matrices for CST elements.

UNIT –III
TWO DIMENSIONAL PROBLEMS

A thin plate of thickness t, with a hole in the middle, is subjected to a uniform traction load, T as shown. This 3-D plate can be analyzed as a **two-dimensional** problem.

2-D problems generally fall into two categories: *plane stress* and *plane strain*.



A *plane stress* problem

a) Plane Stress

The thin plate can be analyzed as a *plane stress* problem, where the normal and shear stresses perpendicular to the *x-y* plane are *assumed* to be zero, i.e.

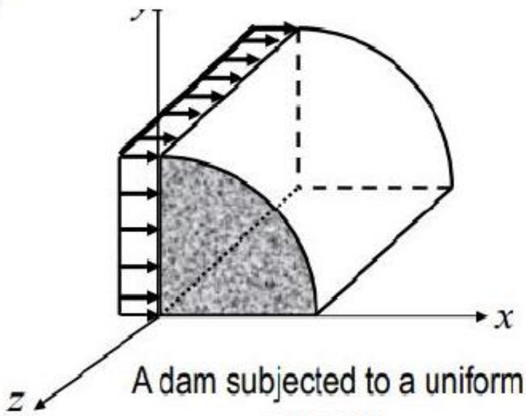
$$\sigma_z = 0;\ \tau_{xz} = 0;\ \tau_{yz} = 0$$

The *nonzero* stress components are

$$\sigma_x \neq 0;\ \sigma_y \neq 0;\ \tau_{xy} \neq 0$$

b) Plane Strain

A dam subjected to uniform pressure and a pipe under a uniform internal pressure can be analyzed in two-dimension as *plain strain* problems.



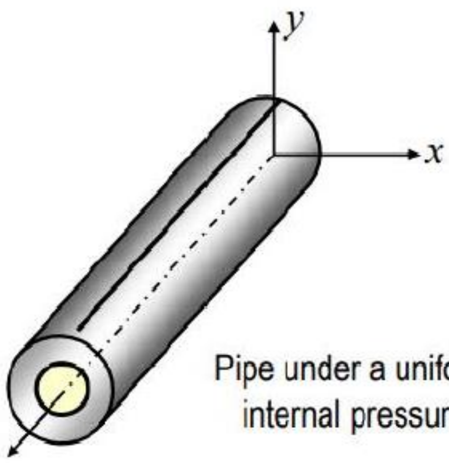
A dam subjected to a uniform pressure

The strain components perpendicular to the *x-y* plane are assumed to be zero, i.e.

$$\varepsilon_z = 0;\ \gamma_{xz} = 0;\ \gamma_{yz} = 0$$

Thus, the *nonzero* strain components are ε_x, ε_y, and γ_{xy}.

$$\varepsilon_x \neq 0;\ \varepsilon_y \neq 0;\ \gamma_{xy} \neq 0$$



Pipe under a uniform internal pressure

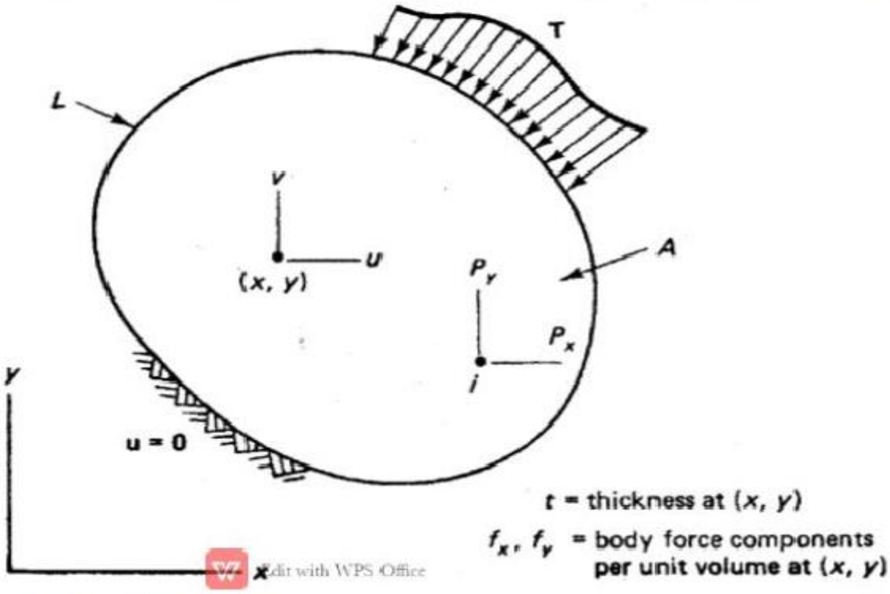
8-2 General Loading Condition

A two-dimensional body can be subjected to **three** types of forces:

a) Concentrated forces, P_x & P_y at a point, i;

b) Body forces, $f_{b,x}$ & $f_{b,y}$ acting at its *centroid*;

c) Traction force, T (i.e. force per unit length), acting along a *perimeter*

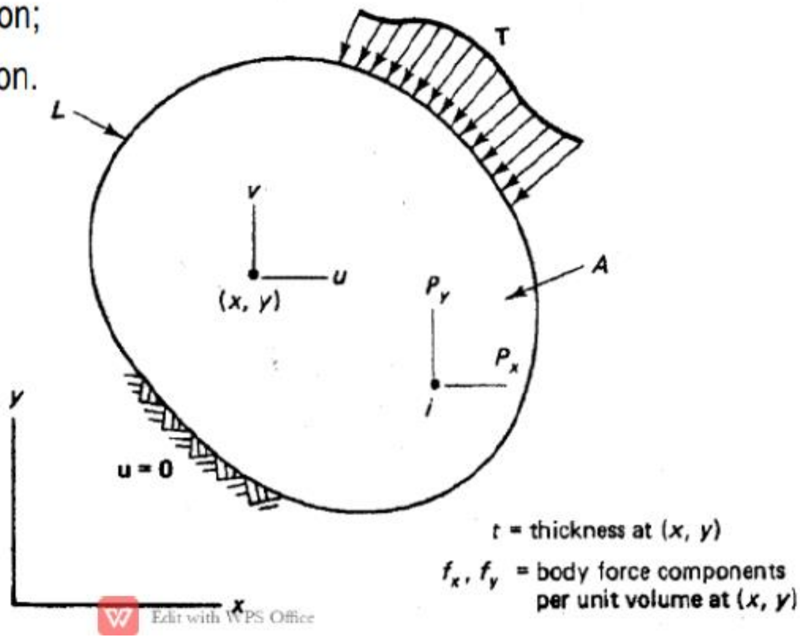


The 2-dimensional body experiences a deformation due to the applied loads.

At any point in the body, there are two components of displacement, i.e.

u = displacement in x-direction;

v = displacement in y-direction.



Stress-Strain Relation

Recall, at any point in the body, there are three components of strains, i.e.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

The corresponding stress components at that point are

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

The stresses and strains are related through,

$$\{\sigma\} = [D]\{\varepsilon\}$$

where $[D]$ is called the *material matrix*, given by

$$[D] = \frac{E}{1-\nu^2} \cdot \begin{Bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{Bmatrix}$$

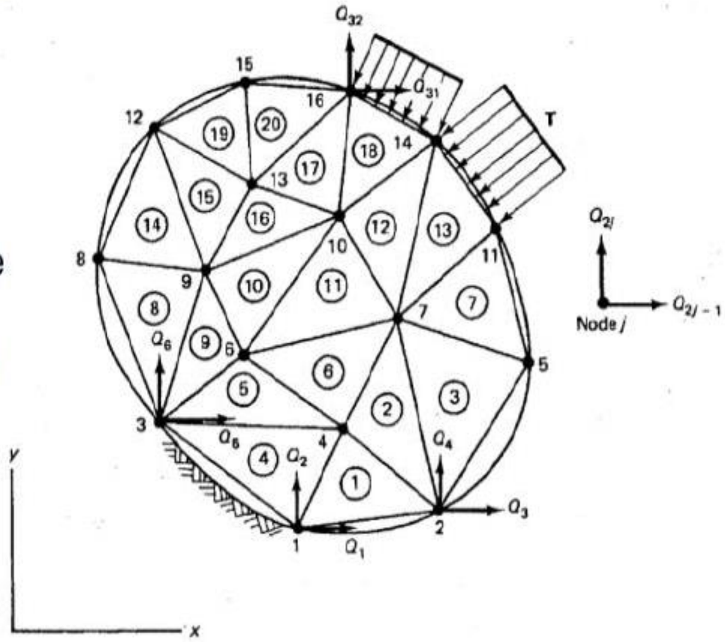
for *plane stress* problems and

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \cdot \begin{Bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{Bmatrix}$$

for *plane strain* problems.

8-3 Finite Element Modeling

The two-dimensional body is transformed into finite element model by subdividing it using triangular elements.



Note:

1. *Unfilled* region exists for curved boundaries, affecting accuracy of the solution. The accuracy can be improved by using smaller elements.

2. There are **two** displacement components at a node. Thus, at a node j, the displacements are:

Q_{2j-1} in x-direction

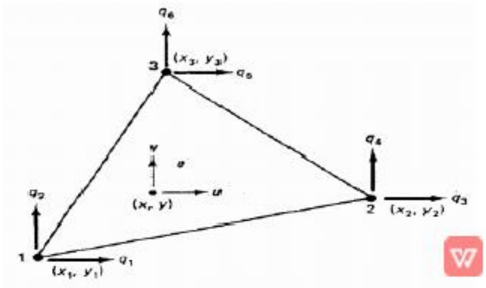
Q_{2j} in y-direction

Difference B/W CST & LST elements

CST

- CST - Constant Strain Triangle
- 3 nodes per Triangle
- First order Triangle Element
- Strain in the element won't vary. Through out the element surface **constant strain** is observed.
- Displacement function is Linear
- Hence the displacement model is

$$\begin{cases} u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y \\ v(x,y) = \beta_1 + \beta_2 x + \beta_3 y \end{cases}$$

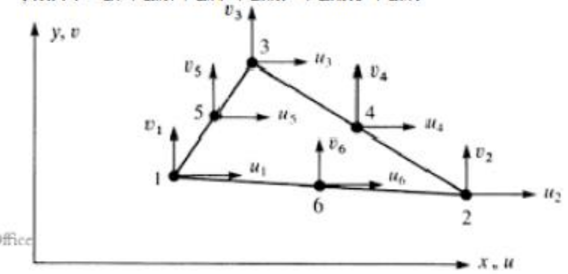


LST

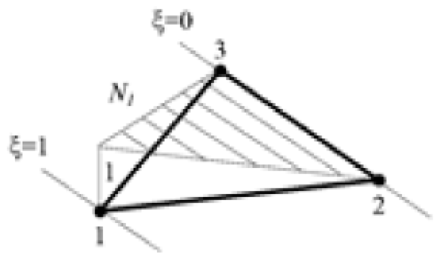
- LST - Linear Strain Triangle
- 6 nodes in Triangle
- Second order Triangle Element
- Strain will vary in the element as **Linear**
- Displacement function is quadratic
- The variation of the displacements over the element may be expressed as:

$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

$$v(x,y) = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$$

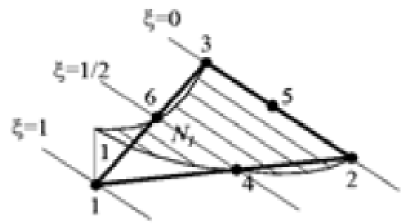


- CST elements are poor in Capturing the bending behaviour
- For same number of elements, true displacement and stresses not obtained in CST elements
- Fig below shows the variation of shape function N1 for the CST element



Shape Function N_1 for CST

- LST elements are good in Capturing the bending behaviour
- For same number of elements, true displacement and stresses obtained better in LST elements
- Fig below shows the variation of shape function N1 for the LST element



Shape Function N_1 for LST

Example 5.1

Evaluate the shape functions N_1, N_2, and N_3 at the interior point P for the triangular element shown in Fig. E5.1.

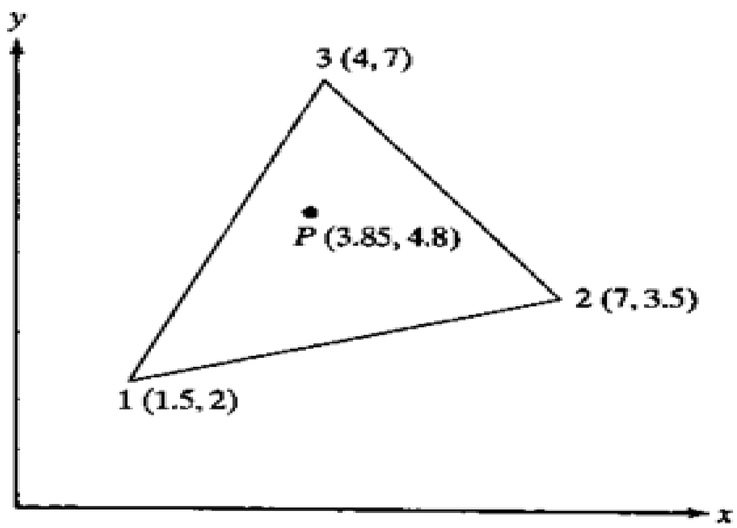


FIGURE E5.1 Examples 5.1 and 5.2.

Solution Using the isoparametric representation (Eqs. 5.15), we have

$$3.85 = 1.5N_1 + 7N_2 + 4N_3 = -2.5\xi + 3\eta + 4$$
$$4.8 = 2N_1 + 3.5N_2 + 7N_3 = -5\xi - 3.5\eta + 7$$

These two equations are rearranged in the form

$$2.5\xi - 3\eta = 0.15$$

$$5\xi + 3.5\eta = 2.2$$

Solving the equations, we obtain $\xi = 0.3$ and $\eta = 0.2$, which implies that

$$N_1 = 0.3 \qquad N_2 = 0.2 \qquad N_3 = 0.5 \qquad \blacksquare$$

In evaluating the strains, partial derivatives of u and v are to be taken with respect to x and y. From Eqs. 5.12 and 5.15, we see that u, v and x, y are functions of ξ and η. That is, $u = u(x(\xi, \eta), y(\xi, \eta))$ and similarly $v = v(x(\xi, \eta), y(\xi, \eta))$. Using the chain rule for partial derivatives of u, we have

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

which can be written in matrix notation as

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} \tag{5.16}$$

where the (2×2) square matrix is denoted as the *Jacobian* of the transformation, $\mathbf{J}$:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{5.17}$$

Some additional properties of the Jacobian are given in the appendix. On taking the derivative of x and y,

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \tag{5.18}$$

Also, from Eq. 5.16,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{5.19}$$

where $\mathbf{J}^{-1}$ is the inverse of the Jacobian $\mathbf{J}$, given by

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \tag{5.20}$$

$$\det \mathbf{J} = x_{13}y_{23} - x_{23}y_{13} \tag{5.21}$$

Example 5.2

Determine the Jacobian of the transformation **J** for the triangular element shown in Fig. E5.1.

Solution We have

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} -2.5 & -5.0 \\ 3.0 & -3.5 \end{bmatrix}$$

Thus, det **J** = 23.75 units. This is twice the area of the triangle. If 1, 2, 3 are in a clockwise order, then det **J** will be negative. ■

From Eqs. 5.19 and 5.20, it follows that

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}\dfrac{\partial u}{\partial \xi} - y_{13}\dfrac{\partial u}{\partial \eta} \\ -x_{23}\dfrac{\partial u}{\partial \xi} + x_{13}\dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{5.23a}$$

Replacing u by the displacement v, we get a similar expression

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}\dfrac{\partial v}{\partial \xi} - y_{13}\dfrac{\partial v}{\partial \eta} \\ -x_{23}\dfrac{\partial v}{\partial \xi} + x_{13}\dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{5.23b}$$

Using the strain–displacement relations (5.5) and Eqs. 5.12b and 5.23, we get

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$= \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}(q_1 - q_5) - y_{13}(q_3 - q_5) \\ -x_{23}(q_2 - q_6) + x_{13}(q_4 - q_6) \\ -x_{23}(q_1 - q_5) + x_{13}(q_3 - q_5) + y_{23}(q_2 - q_6) - y_{13}(q_4 - q_6) \end{Bmatrix} \tag{5.24a}$$

From the definition of x_{ij} and y_{ij}, we can write $y_{31} = -y_{13}$ and $y_{12} = y_{13} - y_{23}$, and so on. The foregoing equation can be written in the form

$$\boldsymbol{\epsilon} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}q_1 + y_{31}q_3 + y_{12}q_5 \\ x_{32}q_2 + x_{13}q_4 + x_{21}q_6 \\ x_{32}q_1 + y_{23}q_2 + x_{13}q_3 + y_{31}q_4 + x_{21}q_5 + y_{12}q_6 \end{Bmatrix} \tag{5.24b}$$

This equation can be written in matrix form as

$$\boldsymbol{\epsilon} = \mathbf{Bq} \tag{5.25}$$

where **B** is a (3×6) element strain–displacement matrix relating the three strains to the six nodal displacements and is given by

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{5.26}$$

It may be noted that all the elements of the **B** matrix are constants expressed in terms of the nodal coordinates.

Example 5.3

Find the strain–nodal displacement matrices $\mathbf{B}^e$ for the elements shown in Fig. E5.3. Use local numbers given at the corners.

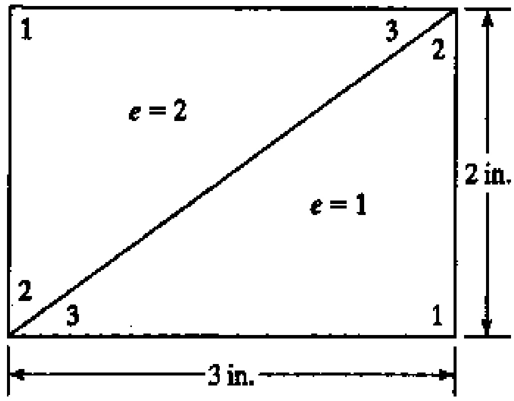


FIGURE E5.3

Solution We have

$$\mathbf{B}^1 = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

where det $\mathbf{J}$ is obtained from $x_{13}y_{23} - x_{23}y_{13} = (3)(2) - (3)(0) = 6$. Using the local numbers at the corners, $\mathbf{B}^2$ can be written using the relationship as

$$\mathbf{B}^2 = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}$$ ■

Potential-Energy Approach

The potential energy of the system, Π, is given by

$$\Pi = \frac{1}{2}\int_A \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}t\,dA - \int_A \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}t\,d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{5.27}$$

In the last term in Eq. 5.27, i indicates the point of application of a point load $\mathbf{P}_i$ and $\mathbf{P}_i = [P_x, P_y]_i^{\mathrm{T}}$. The summation in i gives the potential energy due to all point loads.

Using the triangulation shown in Fig. 5.2, the total potential energy can be written in the form

$$\Pi = \sum_e \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}t\,dA - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}t\,d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{5.28a}$$

or

$$\Pi = \sum_e U_e - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA - \sum \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}t\,d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{5.28b}$$

where $U_e = \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}t\,dA$ is the element strain energy.

THE FOUR-NODE QUADRILATERAL

Consider the general quadrilateral element shown in Fig. 7.1. The local nodes are numbered as 1, 2, 3, and 4 in a *counterclockwise* fashion as shown, and (x_i, y_i) are the coordinates of node i. The vector $\mathbf{q} = [q_1, q_2, \ldots, q_8]^T$ denotes the element displacement vector. The displacement of an interior point P located at (x, y) is represented as $\mathbf{u} = [u(x, y), v(x, y)]^T$.

Shape Functions

Following the steps in earlier chapters, we first develop the shape functions on a master element, shown in Fig. 7.2. The master element is defined in ξ-, η-coordinates (or *natural* coordinates) and is square shaped. The Lagrange shape functions where $i = 1, 2, 3$, and 4, are defined such that N_i is equal to unity at node i and is zero at other nodes. In particular, consider the definition of N_i:

$$\begin{aligned} N_1 &= 1 \quad \text{at node 1} \\ &= 0 \quad \text{at nodes 2, 3, and 4} \end{aligned} \tag{7.1}$$

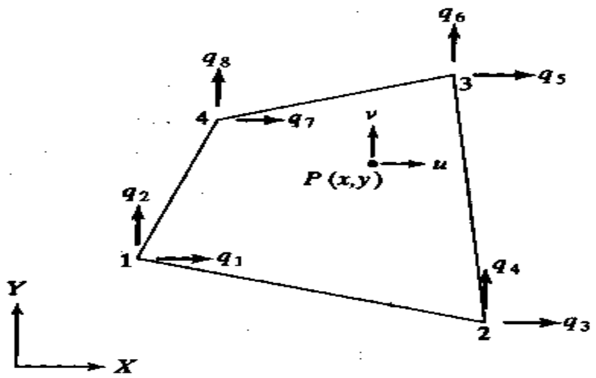


FIGURE 7.1 Four-node quadrilateral element.

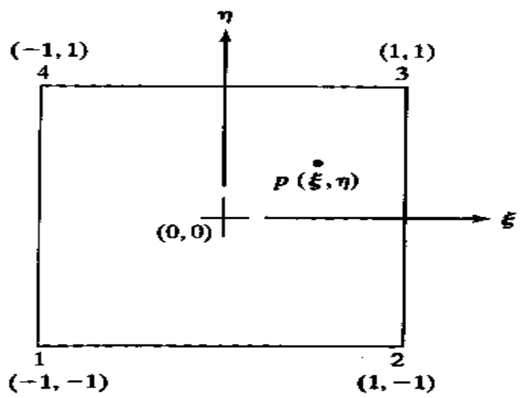


FIGURE 7.2 The quadrilateral element in ξ, η space (the *master* element).

Now, the requirement that $N_1 = 0$ at nodes 2, 3, and 4 is equivalent to requiring that $N_1 = 0$ along edges $\xi = +1$ and $\eta = +1$ (Fig. 7.2). Thus, N_1 has to be of the form

$$N_1 = c(1 - \xi)(1 - \eta) \tag{7.2}$$

where c is some constant. The constant is determined from the condition $N_1 = 1$ at node 1. Since $\xi = -1$, $\eta = -1$ at node 1, we have

$$1 = c(2)(2) \tag{7.3}$$

which yields $c = \frac{1}{4}$. Thus,

$$N_1 = \tfrac{1}{4}(1 - \xi)(1 - \eta) \tag{7.4}$$

All the four shape functions can be written as

$$
\begin{aligned}
N_1 &= \tfrac{1}{4}(1-\xi)(1-\eta) \\
N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta) \\
N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta) \\
N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta)
\end{aligned}
\tag{7.5}
$$

While implementing in a computer program, the compact representation of Eqs. 7.5 is useful

$$N_i = \tfrac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \tag{7.6}$$

where (ξ_i, η_i) are the coordinates of node i.

We now express the displacement field within the element in terms of the nodal values. Thus, if $\mathbf{u} = [u, v]^T$ represents the displacement components of a point located at (ξ, η), and $\mathbf{q}$, dimension (8×1), is the element displacement vector, then

$$
\begin{aligned}
u &= N_1q_1 + N_2q_3 + N_3q_5 + N_4q_7 \\
v &= N_1q_2 + N_2q_4 + N_3q_6 + N_4q_8
\end{aligned}
\tag{7.7a}
$$

which can be written in matrix form as

$$\mathbf{u} = \mathbf{Nq} \tag{7.7b}$$

where

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \tag{7.8}$$

In the isoparametric formulation, we use the *same* shape functions N_i to also express the coordinates of a point within the element in terms of nodal coordinates. Thus,

$$
\begin{aligned}
x &= N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 \\
y &= N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4
\end{aligned}
\tag{7.9}
$$

Subsequently, we will need to express the derivatives of a function in x-, y-coordinates in terms of its derivatives in ξ-, η-coordinates. This is done as follows: A function $f = f(x, y)$, in view of Eqs. 7.9, can be considered to be an implicit function of ξ and η as $f = f[x(\xi, \eta), y(\xi, \eta)]$. Using the chain rule of differentiation, we have

$$
\begin{aligned}
\frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} \\
\frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}
\end{aligned}
\tag{7.10}
$$

or

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} \tag{7.11}$$

where **J** is the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{7.12}$$

In view of Eqs. 7.5 and 7.9, we have

$$\mathbf{J} = \frac{1}{4}\left[\begin{array}{c|c} -(1-\eta)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4 & -(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4 \\ -(1-\xi)x_1-(1+\xi)x_2+(1+\xi)x_3+(1-\xi)x_4 & -(1-\xi)y_1-(1+\xi)y_2+(1+\xi)y_3+(1-\xi)y_4 \end{array}\right] \tag{7.13a}$$

$$\equiv \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{7.13b}$$

Equation 7.11 can be inverted as

$$\begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} \tag{7.14a}$$

or

$$\begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} \tag{7.14b}$$

These expressions will be used in the derivation of the element stiffness matrix.

An additional result that will be needed is the relation

$$dx\,dy = \det \mathbf{J}\, d\xi\, d\eta \tag{7.15}$$

The proof of this result, found in many textbooks on calculus, is given in the appendix.

Element Stiffness Matrix

The stiffness matrix for the quadrilateral element can be derived from the strain energy in the body, given by

$$U = \int_V \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dV \tag{7.16}$$

or

$$U = \sum_e t_e \int_e \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dA \tag{7.17}$$

where t_e is the thickness of element e.

The strain–displacement relations are

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{7.18}$$

By considering $f \equiv u$ in Eq. 7.14b, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{7.19a}$$

Similarly,

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{7.19b}$$

Equations 7.18 and 7.19a,b yield

$$\boldsymbol{\epsilon} = \mathbf{A} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{7.20}$$

where $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \tag{7.21}$$

Now, from the interpolation equations Eqs. 7.7a, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \mathbf{G}\mathbf{q} \tag{7.22}$$

where

$$\mathbf{G} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \tag{7.23}$$

Equations 7.20 and 7.22 now yield

$$\boxed{\boldsymbol{\epsilon} = \mathbf{Bq}} \tag{7.24}$$

where

$$\mathbf{B} = \mathbf{AG} \tag{7.25}$$

The relation $\boldsymbol{\epsilon} = \mathbf{Bq}$ is the desired result. The strain in the element is expressed in terms of its nodal displacement. The stress is now given by

$$\boxed{\boldsymbol{\sigma} = \mathbf{DBq}} \tag{7.26}$$

where $\mathbf{D}$ is a (3×3) material matrix. The strain energy in Eq. 7.17 becomes

$$U = \sum_e \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\left[t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{DB}\det\mathbf{J}\,d\xi\,d\eta\right]\mathbf{q} \tag{7.27a}$$

$$= \sum_e \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q} \tag{7.27b}$$

where

$$\boxed{\mathbf{k}^e = t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{DB}\det\mathbf{J}\,d\xi\,d\eta} \tag{7.28}$$

is the element stiffness matrix of dimension (8×8).

We note here that quantities $\mathbf{B}$ and $\det \mathbf{J}$ in the integral in Eq. (7.28) are involved functions of ξ and η, and so the integration has to be performed numerically. Methods of numerical integration are discussed subsequently.

Element Force Vectors

Body Force A body force that is distributed force per unit volume, contributes to the global load vector $\mathbf{F}$. This contribution can be determined by considering the body force term in the potential-energy expression

$$\int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\,dV \tag{7.29}$$

Using $\mathbf{u} = \mathbf{Nq}$, and treating the body force $\mathbf{f} = [f_x, f_y]^{\mathrm{T}}$ as constant within each element, we get

Axi-Symmetric Models

Elasticity Equations

Elasticity equations are used for solving structural mechanics problems. These equations must be satisfied if an exact solution to a structural mechanics problem is to be obtained. The types of elasticity equations are

➢ 1. Strian – Displacement relationship equations

$$e_x = \frac{\partial u}{\partial x};\ e_y = \frac{\partial v}{\partial y};\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x};\ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x};$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

e_x – Strain in X direction, e_y – Strain in Y direction.

γ_{xy} - Shear Strain in XY plane, γ_{xz} - Shear Strain in XZ plane,

γ_{yz} - Shear Strain in YZ plane

2. Sterss – Strain relationship equation

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} (1-v) & v & v & 0 & 0 & 0 \\ v & (1-v) & v & 0 & 0 & 0 \\ v & v & (1-v) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_x \\ e_x \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$\left\lfloor \qquad\qquad\qquad\qquad\qquad\qquad 2 \right\rfloor$$

σ – Stress, τ – Shear Stress, E – Young's Modulus, v – Poisson's Ratio,

e – Strain, γ - Shear Strain.

3. Equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0\,; \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} + B_y = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + B_z = 0$$

σ – Stress, τ – Shear Stress, B_x - Body force at X direction,

B_y - Body force at Y direction, B_z - Body force at Z direction.

4. Compatibility equations

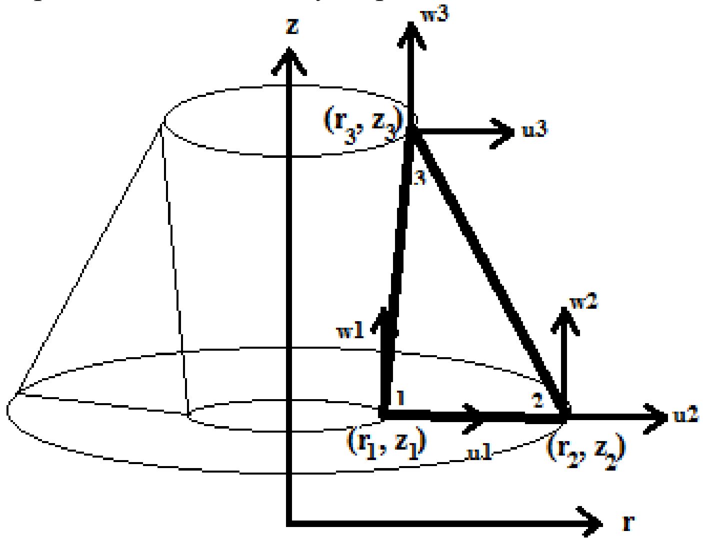
There are six independent compatibility equations, one of which is

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.$$

The other five equations are similarly second order relations.

- **Axisymmetric Elements**

Most of the three-dimensional problems are symmetry about an axis of rotation. Those types of problems are solved by a special two-dimensional element



calledas axisymmetric element.

- **Axisymmetric Formulation**

  The displacement vector u is given by

$$u(r,z) = \begin{Bmatrix} u \\ w \end{Bmatrix}$$

  The stress σ is given by

$$Stress, \{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix}$$

  The strain e is given by

$$Strain, \{e\} = \begin{Bmatrix} e_r \\ e^\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix}$$

- **Equation of shape function for Axisymmetric element**

  Shape function,

$$N_1 = \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A}; \; N_2 = \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A}; \; N_3 = \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A}$$

$\alpha_1 = r_2z_3 - r_3z_2$; $\alpha_2 = r_3z_1 - r_1z_3$; $\alpha_3 = r_1z_2 - r_2z_1$

$\beta_1 = z_2 - z_3$; $\beta_2 = z_3 - z_1$; $\beta_3 = z_1 - z_2$

$\gamma_1 = r_3 - r_2$; $\gamma_2 = r_1 - r_3$; $\gamma_3 = r_2 - r_1$

$2A = (r_2z_3 - r_3z_2) - r_1(r_3z_1 - r_1z_3) + z_1(r_1z_2 - r_2z_1)$

- **Equation of Strain – Displacement Matrix [B] for Axisymmetric element**

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$$r = \frac{r1 + r2 + r3}{3}$$

- **Equation of Stress – Strain Matrix [D] for Axisymmetric element**

$$[D] = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

- **Equation of Stiffness Matrix [K] for Axisymmetric element**

$$[K] = 2\prod rA[B]^T[D][B]$$

$r = \frac{r1 + r2 + r3}{3}$; A = (½) bxh

- **Temperature Effects**

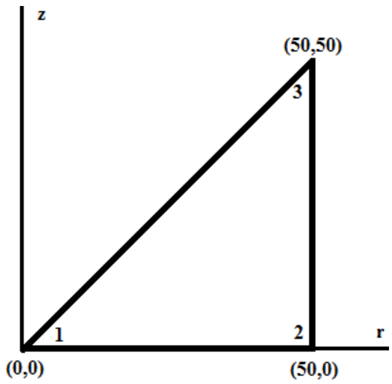
  The thermal force vector is given by

$$\{f\}_t = 2\prod rA[B][D]\{e\}_t$$

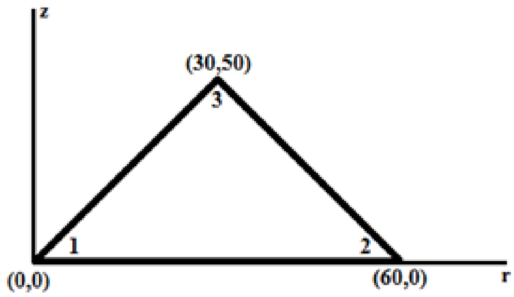
$$\{f\}_t = \begin{Bmatrix} F_1 u \\ F_1 w \\ F_2 u \\ F_2 w \\ F_3 u \\ F_3 w \end{Bmatrix}$$

- **Problem (I set)**

1. For the given element, determine the stiffness matrix. Take E=200GPa and υ= 0.25.



2. For the figure, determine the element stresses. Take E=2.1×10^5N/mm^2 and υ= 0.25. The co – ordinates are in mm. The nodal displacements are u_1=0.05mm, w_1=0.03mm, u_2=0.02mm, w_2=0.02mm, u_3=0.0mm, w_3=0.0mm.



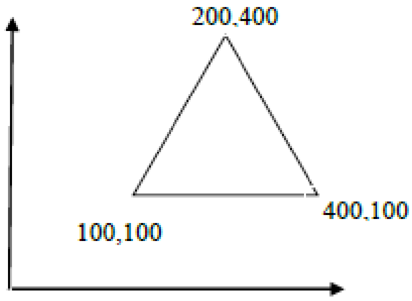
3. A long hollow cylinder of inside diameter 100mm and outside diameter 140mm is subjected to an internal pressure of 4N/mm2. By using two elements on the 15mm length, calculate the displacements at the inner radius.

UNIT III
Tutorial Questions

1. Derive the shape functions for CST Element

2. Derive the strain displacement matrix for CST element.

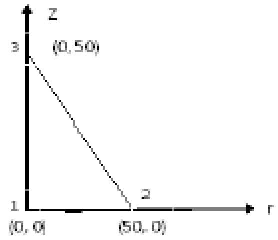
3. For the plane stress element shown in figure the nodal displacements are U1= 2.0mm, V1=1.0mm U2= 1.0 mm,V2= 1.5mm, U3= 2.5mm,V3=0.5mm, Take E= 210GPa, v= 0.25, t=10mm. Determine the strain-Displacement matrix [B]



Assignment Questions

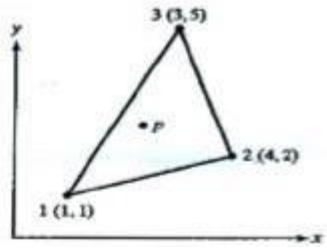
1. For axisymmetric element shown in figure, determine the strain-displacement matrix.

   Let $E = 2.1\times10^5 N/mm^2$ and $\nu = 0.25$. The co-ordinates shown in figure are in millimeters.



2. a) Write the difference between CST and LST elements

   b) For point P located inside the triangle shown in the figure below the shape functions N1 and N2 are 0.15 and 0.25, respectively. Determine the x and y coordinates of point P.



3. Derive the shape functions for axisymmetric element

UNIT 4

ISO-PARAMETRIC FORMULATION
&
HEAT TRANSFER PROBLEMS

Syllabus

Iso-Parametric Formulation: Concepts, sub parametric, super parametric elements, 2 dimensional 4 nodded iso-parametric elements, and numerical integration. Heat Transfer Problems: One dimensional steady state analysis composite wall. One dimensional fin analysis and two-dimensional analysis of thin plate.

OBJECTIVE:

To learn the application of FEM equations for Iso-Parametric and heat transfer problems

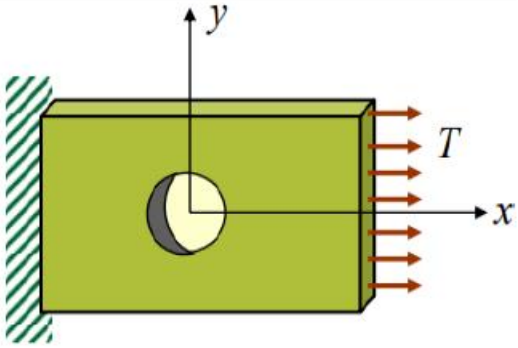
OUTCOME:

Formulate FE Characteristic equations for Isoparamteric problems and heattransfer problem.

UNIT –III
TWO DIMENSIONAL PROBLEMS

A thin plate of thickness t, with a hole in the middle, is subjected to a uniform traction load, T as shown. This 3-D plate can be analyzed as a **two-dimensional** problem.

2-D problems generally fall into two categories: *plane stress* and *plane strain*.



A *plane stress* problem

a) Plane Stress

The thin plate can be analyzed as a *plane stress* problem, where the normal and shear stresses perpendicular to the x-y plane are *assumed* to be zero, i.e.

$$\sigma_z = 0;\ \tau_{xz} = 0;\ \tau_{yz} = 0$$

The *nonzero* stress components are

$$\sigma_x \neq 0;\ \sigma_y \neq 0;\ \tau_{xy} \neq 0$$

b) Plane Strain

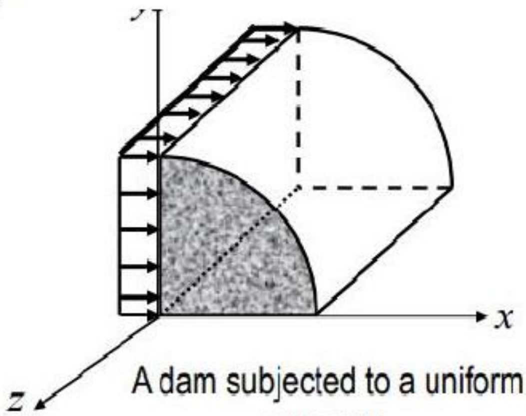
A dam subjected to uniform pressure and a pipe under a uniform internal pressure can be analyzed in two-dimension as *plain strain* problems.

The strain components perpendicular to the x-y plane are assumed to be zero, i.e.

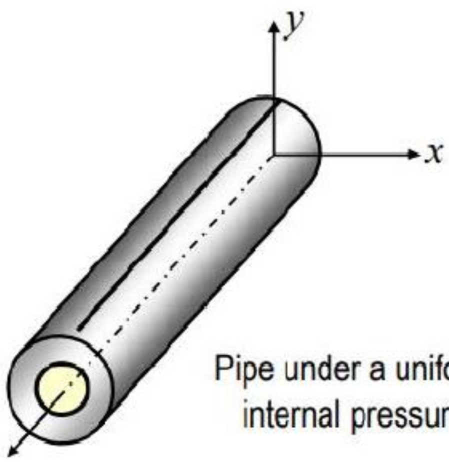
$$\varepsilon_z = 0;\ \gamma_{xz} = 0;\ \gamma_{yz} = 0$$

Thus, the *nonzero* strain components are ε_x, ε_y, and γ_{xy}.

$$\varepsilon_x \neq 0;\ \varepsilon_y \neq 0;\ \gamma_{xy} \neq 0$$



A dam subjected to a uniform pressure



Pipe under a uniform internal pressure

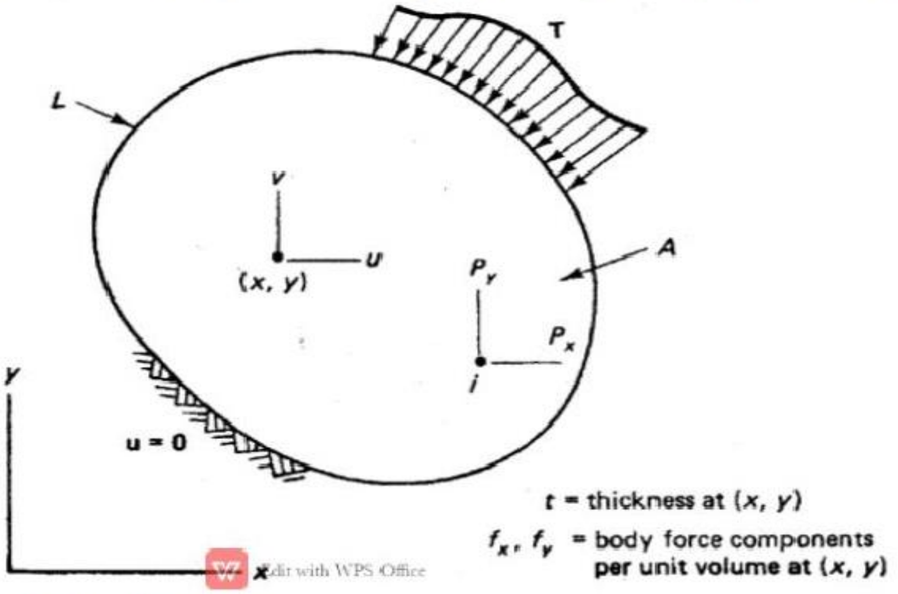
8-2 General Loading Condition

A two-dimensional body can be subjected to **three** types of forces:

a) Concentrated forces, P_x & P_y at a point, i;

b) Body forces, $f_{b,x}$ & $f_{b,y}$ acting at its *centroid*;

c) Traction force, T (i.e. force per unit length), acting along a *perimeter*

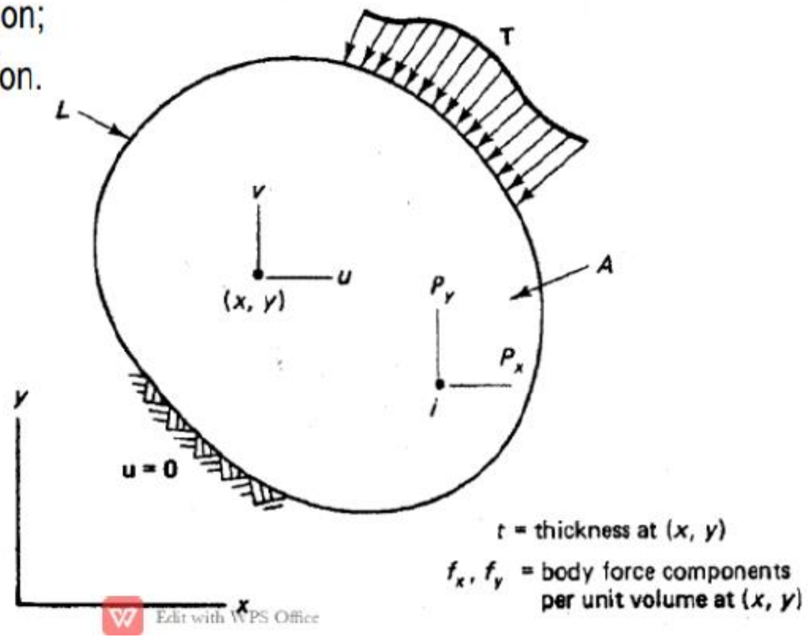


The 2-dimensional body experiences a deformation due to the applied loads.

At any point in the body, there are two components of displacement, i.e.

u = displacement in x-direction;

v = displacement in y-direction.



Stress-Strain Relation

Recall, at any point in the body, there are three components of strains, i.e.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

The corresponding stress components at that point are

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

The stresses and strains are related through,

$$\{\sigma\} = [D]\{\varepsilon\}$$

where $[D]$ is called the *material matrix*, given by

$$[D] = \frac{E}{1-\nu^2} \cdot \begin{Bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{Bmatrix}$$

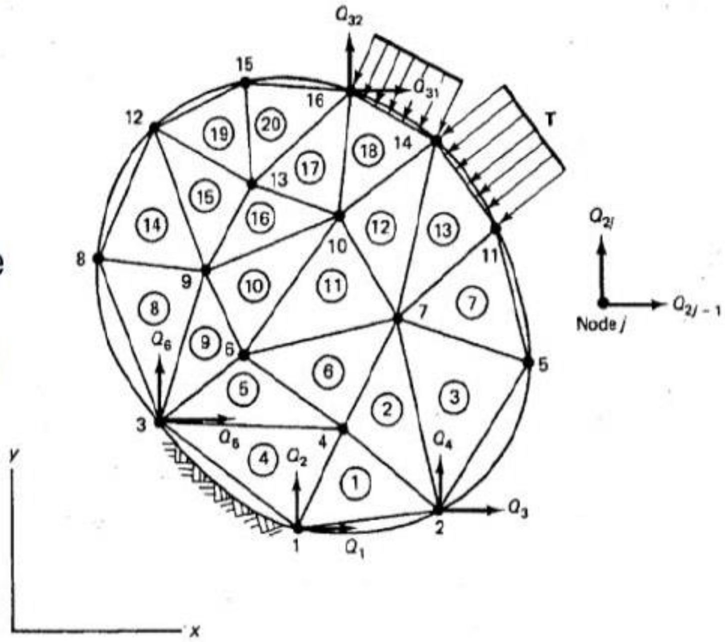
for *plane stress* problems and

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \cdot \begin{Bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{Bmatrix}$$

for *plane strain* problems.

8-3 Finite Element Modeling

The two-dimensional body is transformed into finite element model by subdividing it using triangular elements.



Note:

1. *Unfilled* region exists for curved boundaries, affecting accuracy of the solution. The accuracy can be improved by using smaller elements.

2. There are **two** displacement components at a node. Thus, at a node j, the displacements are:

Q_{2j-1} in x-direction

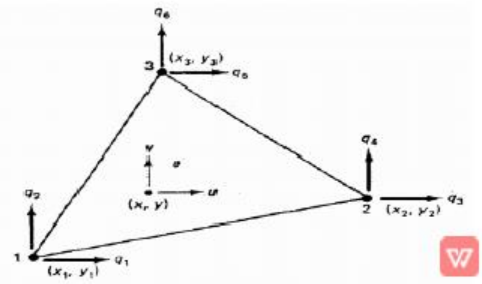
Q_{2j} in y-direction

Difference B/W CST & LST elements

CST

- CST - Constant Strain Triangle
- 3 nodes per Triangle
- First order Triangle Element
- Strain in the element won't vary. Through out the element surface **constant strain** is observed.
- Displacement function is Linear
- Hence the displacement model is

$$\begin{cases} u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y \\ v(x,y) = \beta_1 + \beta_2 x + \beta_3 y \end{cases}$$

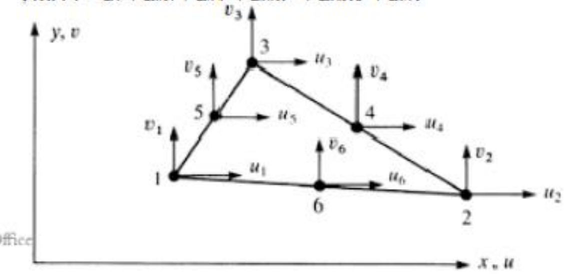


LST

- LST - Linear Strain Triangle
- 6 nodes in Triangle
- Second order Triangle Element
- Strain will vary in the element as **Linear**
- Displacement function is quadratic
- The variation of the displacements over the element may be expressed as:

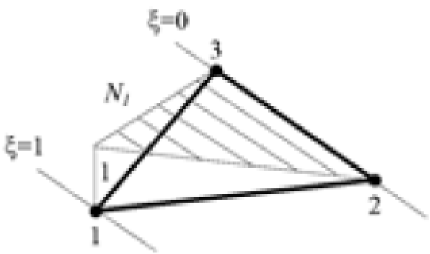
$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

$$v(x,y) = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$$

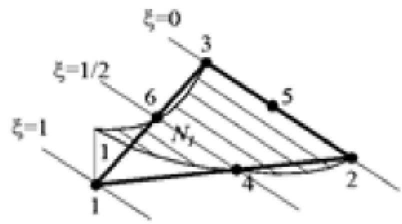


- CST elements are poor in Capturing the bending behaviour
- For same number of elements, true displacement and stresses not obtained in CST elements
- Fig below shows the variation of shape function N1 for the CST element

- LST elements are good in Capturing the bending behaviour
- For same number of elements, true displacement and stresses obtained better in LST elements
- Fig below shows the variation of shape function N1 for the LST element



Shape Function N_1 for CST



Shape Function N_1 for LST

Example 5.1

Evaluate the shape functions N_1, N_2, and N_3 at the interior point P for the triangular element shown in Fig. E5.1.

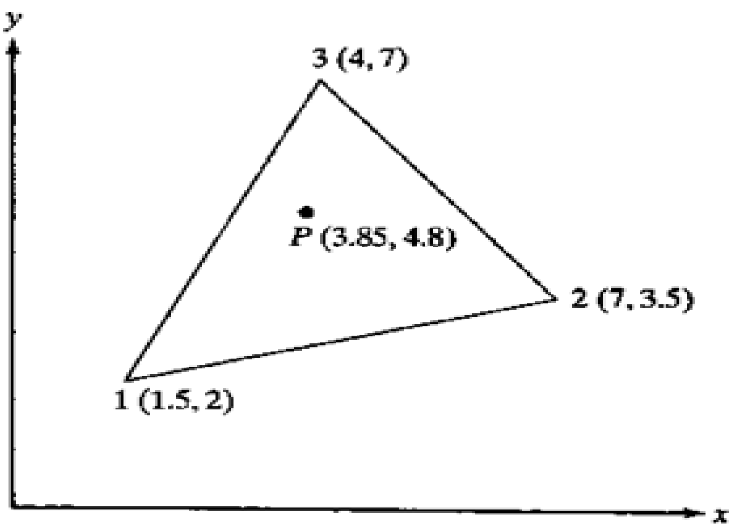


FIGURE E5.1 Examples 5.1 and 5.2.

Solution Using the isoparametric representation (Eqs. 5.15), we have

$$3.85 = 1.5N_1 + 7N_2 + 4N_3 = -2.5\xi + 3\eta + 4$$
$$4.8 = 2N_1 + 3.5N_2 + 7N_3 = -5\xi - 3.5\eta + 7$$

These two equations are rearranged in the form

$$2.5\xi - 3\eta = 0.15$$

$$5\xi + 3.5\eta = 2.2$$

Solving the equations, we obtain $\xi = 0.3$ and $\eta = 0.2$, which implies that

$$N_1 = 0.3 \qquad N_2 = 0.2 \qquad N_3 = 0.5$$ ■

In evaluating the strains, partial derivatives of u and v are to be taken with respect to x and y. From Eqs. 5.12 and 5.15, we see that u, v and x, y are functions of ξ and η. That is, $u = u(x(\xi, \eta), y(\xi, \eta))$ and similarly $v = v(x(\xi, \eta), y(\xi, \eta))$. Using the chain rule for partial derivatives of u, we have

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

which can be written in matrix notation as

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} \tag{5.16}$$

where the (2×2) square matrix is denoted as the *Jacobian* of the transformation, $\mathbf{J}$:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{5.17}$$

Some additional properties of the Jacobian are given in the appendix. On taking the derivative of x and y,

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \tag{5.18}$$

Also, from Eq. 5.16,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{5.19}$$

where $\mathbf{J}^{-1}$ is the inverse of the Jacobian $\mathbf{J}$, given by

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \tag{5.20}$$

$$\det \mathbf{J} = x_{13}y_{23} - x_{23}y_{13} \tag{5.21}$$

Example 5.2

Determine the Jacobian of the transformation **J** for the triangular element shown in Fig. E5.1.

Solution We have

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} -2.5 & -5.0 \\ 3.0 & -3.5 \end{bmatrix}$$

Thus, det **J** = 23.75 units. This is twice the area of the triangle. If 1, 2, 3 are in a clockwise order, then det **J** will be negative. ■

From Eqs. 5.19 and 5.20, it follows that

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}\dfrac{\partial u}{\partial \xi} - y_{13}\dfrac{\partial u}{\partial \eta} \\ -x_{23}\dfrac{\partial u}{\partial \xi} + x_{13}\dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{5.23a}$$

Replacing u by the displacement v, we get a similar expression

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}\dfrac{\partial v}{\partial \xi} - y_{13}\dfrac{\partial v}{\partial \eta} \\ -x_{23}\dfrac{\partial v}{\partial \xi} + x_{13}\dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{5.23b}$$

Using the strain–displacement relations (5.5) and Eqs. 5.12b and 5.23, we get

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$= \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}(q_1 - q_5) - y_{13}(q_3 - q_5) \\ -x_{23}(q_2 - q_6) + x_{13}(q_4 - q_6) \\ -x_{23}(q_1 - q_5) + x_{13}(q_3 - q_5) + y_{23}(q_2 - q_6) - y_{13}(q_4 - q_6) \end{Bmatrix} \tag{5.24a}$$

From the definition of x_{ij} and y_{ij}, we can write $y_{31} = -y_{13}$ and $y_{12} = y_{13} - y_{23}$, and so on. The foregoing equation can be written in the form

$$\boldsymbol{\epsilon} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}q_1 + y_{31}q_3 + y_{12}q_5 \\ x_{32}q_2 + x_{13}q_4 + x_{21}q_6 \\ x_{32}q_1 + y_{23}q_2 + x_{13}q_3 + y_{31}q_4 + x_{21}q_5 + y_{12}q_6 \end{Bmatrix} \tag{5.24b}$$

This equation can be written in matrix form as

$$\boldsymbol{\epsilon} = \mathbf{Bq} \tag{5.25}$$

where **B** is a (3 × 6) element strain–displacement matrix relating the three strains to the six nodal displacements and is given by

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{5.26}$$

It may be noted that all the elements of the **B** matrix are constants expressed in terms of the nodal coordinates.

Example 5.3

Find the strain–nodal displacement matrices $\mathbf{B}^e$ for the elements shown in Fig. E5.3. Use local numbers given at the corners.

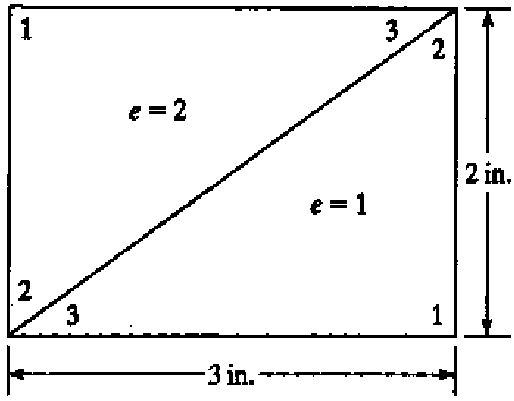


FIGURE E5.3

Solution We have

$$\mathbf{B}^1 = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

where $\det \mathbf{J}$ is obtained from $x_{13}y_{23} - x_{23}y_{13} = (3)(2) - (3)(0) = 6$. Using the local numbers at the corners, $\mathbf{B}^2$ can be written using the relationship as

$$\mathbf{B}^2 = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}$$ ■

Potential-Energy Approach

The potential energy of the system, Π, is given by

$$\Pi = \frac{1}{2}\int_A \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}t\,dA - \int_A \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}t\,d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \qquad (5.27)$$

In the last term in Eq. 5.27, i indicates the point of application of a point load $\mathbf{P}_i$ and $\mathbf{P}_i = [P_x, P_y]_i^{\mathrm{T}}$. The summation in i gives the potential energy due to all point loads.

Using the triangulation shown in Fig. 5.2, the total potential energy can be written in the form

$$\Pi = \sum_e \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}t\,dA - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}t\,d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \qquad (5.28a)$$

or

$$\Pi = \sum_e U_e - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA - \sum \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}t\,d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \qquad (5.28b)$$

where $U_e = \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}t\,dA$ is the element strain energy.

THE FOUR-NODE QUADRILATERAL

Consider the general quadrilateral element shown in Fig. 7.1. The local nodes are numbered as 1, 2, 3, and 4 in a *counterclockwise* fashion as shown, and (x_i, y_i) are the coordinates of node *i*. The vector $\mathbf{q} = [q_1, q_2, \ldots, q_8]^T$ denotes the element displacement vector. The displacement of an interior point *P* located at (x, y) is represented as $\mathbf{u} = [u(x, y), v(x, y)]^T$.

Shape Functions

Following the steps in earlier chapters, we first develop the shape functions on a master element, shown in Fig. 7.2. The master element is defined in ξ-, η-coordinates (or *natural* coordinates) and is square shaped. The Lagrange shape functions where $i = 1, 2, 3,$ and 4, are defined such that N_i is equal to unity at node *i* and is zero at other nodes. In particular, consider the definition of N_i:

$$\begin{aligned} N_1 &= 1 \quad \text{at node 1} \\ &= 0 \quad \text{at nodes 2, 3, and 4} \end{aligned} \tag{7.1}$$

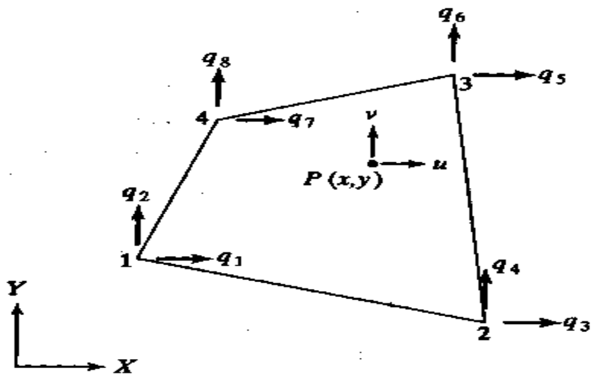


FIGURE 7.1 Four-node quadrilateral element.

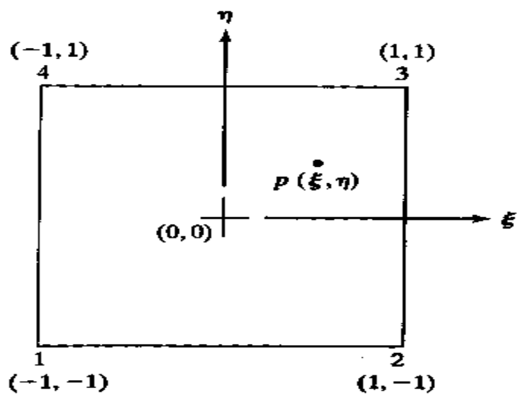


FIGURE 7.2 The quadrilateral element in ξ, η space (the *master* element).

Now, the requirement that $N_1 = 0$ at nodes 2, 3, and 4 is equivalent to requiring that $N_1 = 0$ along edges $\xi = +1$ and $\eta = +1$ (Fig. 7.2). Thus, N_1 has to be of the form

$$N_1 = c(1 - \xi)(1 - \eta) \tag{7.2}$$

where c is some constant. The constant is determined from the condition $N_1 = 1$ at node 1. Since $\xi = -1$, $\eta = -1$ at node 1, we have

$$1 = c(2)(2) \tag{7.3}$$

which yields $c = \frac{1}{4}$. Thus,

$$N_1 = \tfrac{1}{4}(1 - \xi)(1 - \eta) \tag{7.4}$$

All the four shape functions can be written as

$$
\begin{aligned}
N_1 &= \tfrac{1}{4}(1-\xi)(1-\eta)\\
N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta)\\
N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta)\\
N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta)
\end{aligned}
\tag{7.5}
$$

While implementing in a computer program, the compact representation of Eqs. 7.5 is useful

$$N_i = \tfrac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \tag{7.6}$$

where (ξ_i, η_i) are the coordinates of node i.

We now express the displacement field within the element in terms of the nodal values. Thus, if $\mathbf{u} = [u, v]^{\mathrm{T}}$ represents the displacement components of a point located at (ξ, η), and $\mathbf{q}$, dimension (8×1), is the element displacement vector, then

$$
\begin{aligned}
u &= N_1 q_1 + N_2 q_3 + N_3 q_5 + N_4 q_7\\
v &= N_1 q_2 + N_2 q_4 + N_3 q_6 + N_4 q_8
\end{aligned}
\tag{7.7a}
$$

which can be written in matrix form as

$$\mathbf{u} = \mathbf{N}\mathbf{q} \tag{7.7b}$$

where

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \tag{7.8}$$

In the isoparametric formulation, we use the *same* shape functions N_i to also express the coordinates of a point within the element in terms of nodal coordinates. Thus,

$$
\begin{aligned}
x &= N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4\\
y &= N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4
\end{aligned}
\tag{7.9}
$$

Subsequently, we will need to express the derivatives of a function in x-, y-coordinates in terms of its derivatives in ξ-, η-coordinates. This is done as follows: A function $f = f(x, y)$, in view of Eqs. 7.9, can be considered to be an implicit function of ξ and η as $f = f[x(\xi,\eta), y(\xi,\eta)]$. Using the chain rule of differentiation, we have

$$
\begin{aligned}
\frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}\\
\frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}
\end{aligned}
\tag{7.10}
$$

or

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi}\\[2mm] \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial f}{\partial x}\\[2mm] \dfrac{\partial f}{\partial y} \end{Bmatrix} \tag{7.11}$$

where **J** is the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{7.12}$$

In view of Eqs. 7.5 and 7.9, we have

$$\mathbf{J} = \frac{1}{4}\left[\begin{array}{c|c} -(1-\eta)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4 & -(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4 \\ -(1-\xi)x_1-(1+\xi)x_2+(1+\xi)x_3+(1-\xi)x_4 & -(1-\xi)y_1-(1+\xi)y_2+(1+\xi)y_3+(1-\xi)y_4 \end{array}\right] \tag{7.13a}$$

$$\equiv \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{7.13b}$$

Equation 7.11 can be inverted as

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} \tag{7.14a}$$

or

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} \tag{7.14b}$$

These expressions will be used in the derivation of the element stiffness matrix.

An additional result that will be needed is the relation

$$dx\, dy = \det \mathbf{J}\, d\xi\, d\eta \tag{7.15}$$

The proof of this result, found in many textbooks on calculus, is given in the appendix.

Element Stiffness Matrix

The stiffness matrix for the quadrilateral element can be derived from the strain energy in the body, given by

$$U = \int_V \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dV \tag{7.16}$$

or

$$U = \sum_e t_e \int_e \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dA \tag{7.17}$$

where t_e is the thickness of element e.

The strain–displacement relations are

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{7.18}$$

By considering $f \equiv u$ in Eq. 7.14b, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{7.19a}$$

Similarly,

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{7.19b}$$

Equations 7.18 and 7.19a,b yield

$$\boldsymbol{\epsilon} = \mathbf{A} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{7.20}$$

where $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \tag{7.21}$$

Now, from the interpolation equations Eqs. 7.7a, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \mathbf{Gq} \tag{7.22}$$

where

$$\mathbf{G} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \tag{7.23}$$

Equations 7.20 and 7.22 now yield

$$\boxed{\boldsymbol{\epsilon} = \mathbf{Bq}} \tag{7.24}$$

where

$$\mathbf{B} = \mathbf{AG} \tag{7.25}$$

The relation $\boldsymbol{\epsilon} = \mathbf{Bq}$ is the desired result. The strain in the element is expressed in terms of its nodal displacement. The stress is now given by

$$\boxed{\boldsymbol{\sigma} = \mathbf{DBq}} \tag{7.26}$$

where $\mathbf{D}$ is a (3×3) material matrix. The strain energy in Eq. 7.17 becomes

$$U = \sum_e \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\left[t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\det\mathbf{J}\,d\xi\,d\eta\right]\mathbf{q} \tag{7.27a}$$

$$= \sum_e \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q} \tag{7.27b}$$

where

$$\boxed{\mathbf{k}^e = t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\det\mathbf{J}\,d\xi\,d\eta} \tag{7.28}$$

is the element stiffness matrix of dimension (8×8).

We note here that quantities $\mathbf{B}$ and $\det \mathbf{J}$ in the integral in Eq. (7.28) are involved functions of ξ and η, and so the integration has to be performed numerically. Methods of numerical integration are discussed subsequently.

Element Force Vectors

Body Force A body force that is distributed force per unit volume, contributes to the global load vector $\mathbf{F}$. This contribution can be determined by considering the body force term in the potential-energy expression

$$\int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\,dV \tag{7.29}$$

Using $\mathbf{u} = \mathbf{Nq}$, and treating the body force $\mathbf{f} = [f_x, f_y]^{\mathrm{T}}$ as constant within each element, we get

UNIT –IV
ISOPARAMETRICFORMULATION

Definition:

The term ***isoparametric*** (same parameters) is derived from the use of the same shape (interpolation) functions ***N*** to define the element's ***geometric shape*** as are used to define the ***displacements*** within the element.

Alternatively:

The basic principle of isoparametric elements is that the interpolation functions for the displacements are also used to represent the geometry of the element.

$$u = \sum_{i=1}^{4} N_i u_i \quad , \quad v = \sum_{i=1}^{4} N_i v_i$$

$$x = \sum_{i=1}^{4} N_i x_i \quad , \quad y = \sum_{i=1}^{4} N_i y_i$$

Basic Principle of Isoparametric Elements

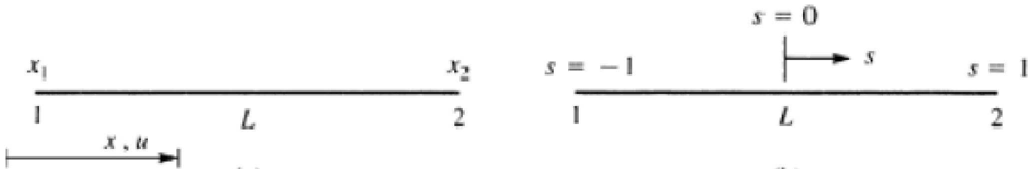
- In this formulation, displacements are expressed in terms of the ***natural*** (local) coordinates and then differentiated with respect to global coordinates. Accordingly, a transformation matrix [J], called ***Jacobian***, is produced.

- If the geometric interpolation functions are of lower order than the displacement shape functions, the element is called ***subparametric***. If the reverse is true, the element is referred to as ***superparametric***.

- The ***isoparametric formulation*** is generally applicable to 1-, 2- and 3- dimensional stress analysis. The isoparametric family includes elements for plane, solid, plate, and shell problems. Also, it is applicable for ***nonstructural*** problems.

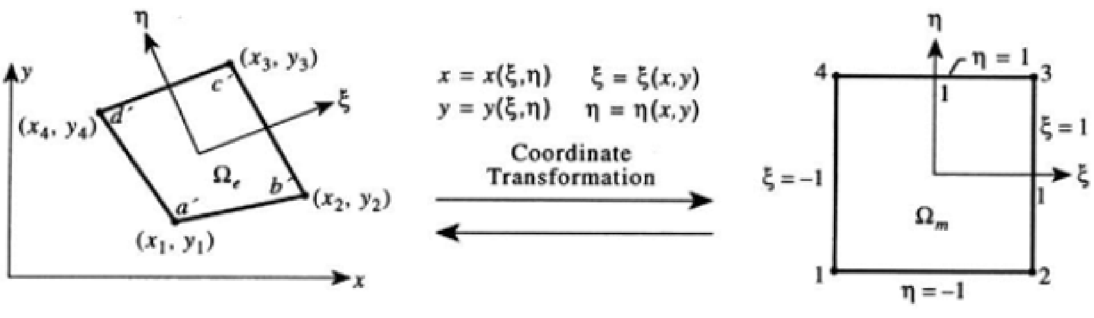
- The isoparametric formulation makes it possible to generate elements that are **nonrectangular** and have **curved** sides. So it can facilitate an accurate representation of irregular elements.

- Numerous **commercial** computer programs have adopted this formulation for their various libraries of elements.



(a) (b)

Two-Noded Bar Isoparametric Element



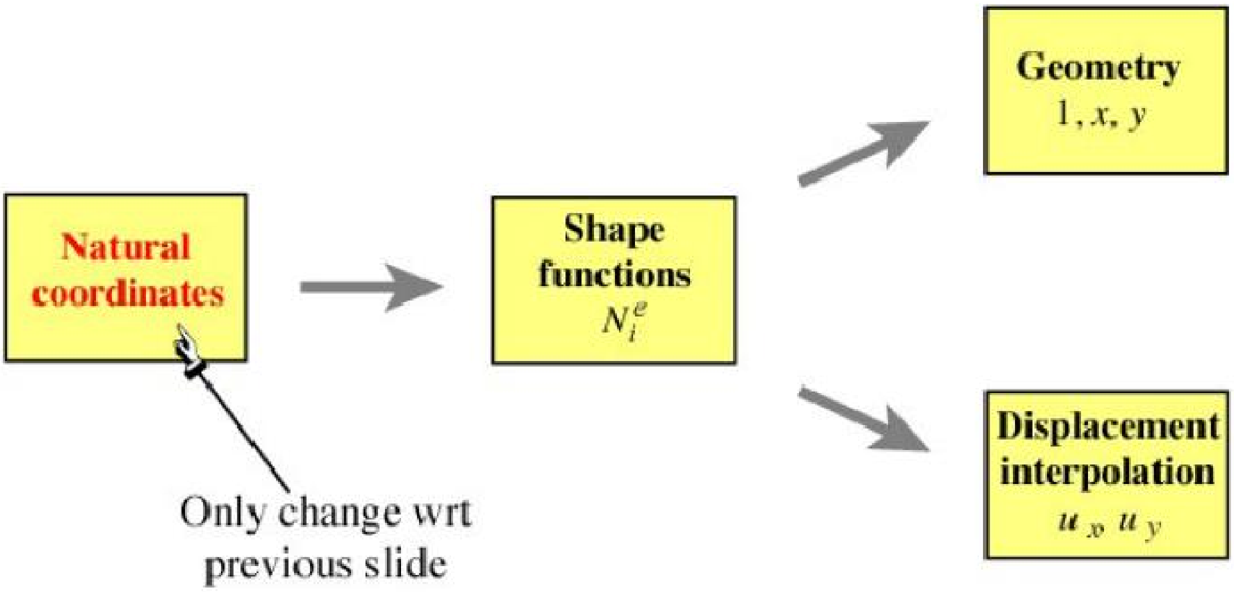
(a) Slave (distorted) element

(b) Master (parent) element

Isoparametric coordinate transformation.

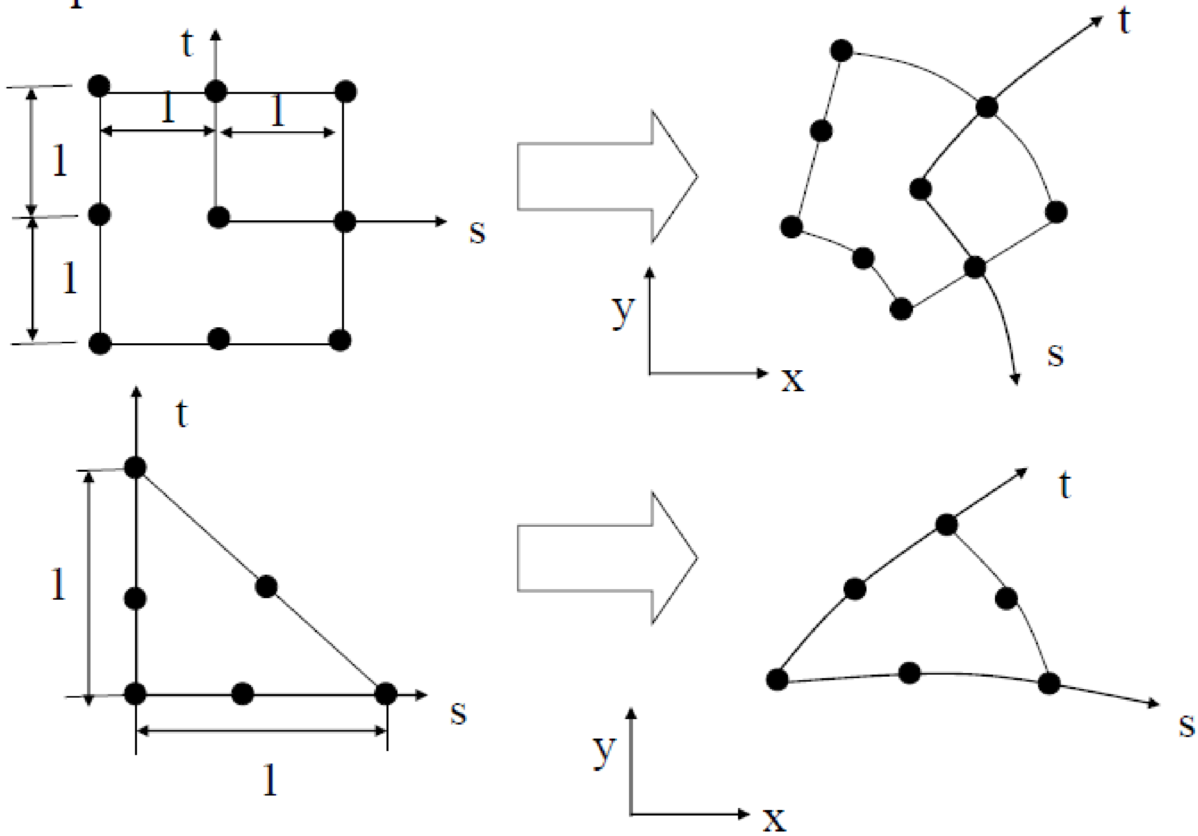
As shown in the figure, the local (natural) coordinate system (ξ, η) for the two elements have their origins at the centroids of the elements, with (ξ, η) varying form –1 to 1. The natural coordinate system needs not to be orthogonal and neither has to be parallel to the x-y axes. The coordinate transformation will map the point (ξ, η) in the master element to $x(\xi, \eta)$ and $y(\xi, \eta)$ in the slave element.

Isoparametric Representation for *any* 2D Element



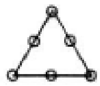
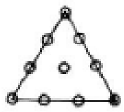
In 3D: $1, x, y$ **becomes** $1, x, y, z$ **etc**

Examples



Step 2: Select Displacement Functions

In other words, we look for shape functions that map the regular shape element in isoparametric coordinates to the quadrilateral in the x-y coordinates whose size and shape are determined by the eight nodal coordinates $x_1, y_1, x_2, y_2, \ldots, x_4, y_4$.

Terms in Pascal Triangle	*Polynomial Degree*	*Number of Terms*	*Triangle*
1	0 (constant)	1	
$x \quad y$	1 (linear)	3	CST 
$x^2 \quad xy \quad y^2$	2 (quadratic)	6	LST 
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3 (cubic)	10	QST 

$$x(\xi,\eta) = a_1 + a_2\,\xi + a_3\,\eta + a_4\,\xi\eta$$

$$y(\xi,\eta) = a_5 + a_6\,\xi + a_7\,\eta + a_8\,\xi\eta$$

$$\begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = \begin{Bmatrix} a_1 + a_2\,\xi + a_3\,\eta + a_4\,\xi\eta \\ a_5 + a_6\,\xi + a_7\,\eta + a_8\,\xi\eta \end{Bmatrix}$$

$$\begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} 1 & \xi & \eta & \xi\eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \xi & \eta & \xi\eta \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$x(\xi,\eta)=[1 \quad \xi \quad \eta \quad \xi\eta]\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}=[1 \quad \xi \quad \eta \quad \xi\eta]\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$=\frac{1}{4}[(1-\xi)(1-\eta)x_1+(1+\xi)(1-\eta)x_2+(1+\xi)(1+\eta)x_3+(1-\xi)(1+\eta)x_4]$$

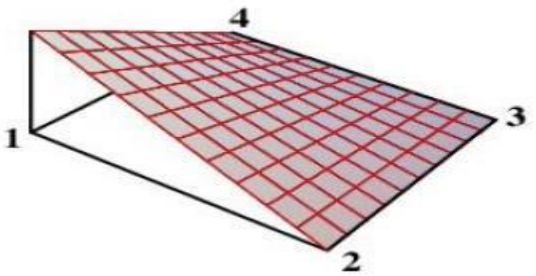
$$\begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix}=\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}\begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}=\begin{Bmatrix} \sum_{i=1}^{4} N_i \, x_i \\ \sum_{i=1}^{4} N_i \, y_i \end{Bmatrix}$$

Shape Function for 4-Nodes quadrilateral Elements

$$N_1=\frac{1}{4}(1-\xi)(1-\eta) \quad , \quad N_2=\frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3=\frac{1}{4}(1+\xi)(1+\eta) \quad , \quad N_4=\frac{1}{4}(1-\xi)(1+\eta)$$

These shape functions are seen to map the *(ξ,η)* coordinates of any point in the rectangular element in the above master element to ***x and y*** coordinates in the quadrilateral (slave) element.
For example, consider the coordinates of node 1, where:
$\xi=-1, \eta=-1$ using the above equation, we get $x=x_1$, $y=y_1$



$$N_1^e \quad = \tfrac{1}{4}(1-\xi)(1-\eta)$$

Shape Function for 4-Nodes quadrilateral Elements

$$N_i = \begin{cases} 1 \; at \; node \, i \\ 0 \; at \; all \; other \; nodes \end{cases}$$

$$\sum_{i=1}^{n} N_i = 1 \quad , \qquad (i = 1, 2, , \cdots, n)$$

where n = the number of shape functions associated with number of nodes

$$\begin{Bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{4} N_i \, u_i \\ \sum_{i=1}^{4} N_i \, v_i \end{Bmatrix}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [N][d]$$

where u *and* v are displacements parallel to the global x and y coordinates

Step 3: Define the Strain/Displacement and Stress/Strain Relationships

Using Chain Rule

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix}$$

Can be computed ⟸

$$\begin{Bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{Bmatrix}$$

⟸ We want to compute these for the B matrix

This is known as the **Jacobian** matrix (J) for the mapping $(\xi, \eta) \rightarrow (x, y)$

$$\begin{Bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$[J]=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}x_i & \sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}y_i\\ \sum_{i=1}^{4}\frac{\partial N_i}{\partial \eta}x_i & \sum_{i=1}^{4}\frac{\partial N_i}{\partial \eta}y_i\end{bmatrix}$$

$$[J]^{-1}=\frac{1}{|J|}\begin{bmatrix}\frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi}\\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi}\end{bmatrix}$$

where

$$|J|=\frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta}-\frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi}$$

Since:

$$\begin{Bmatrix}\frac{\partial N}{\partial x}\\ \frac{\partial N}{\partial y}\end{Bmatrix}=[J]^{-1}\begin{Bmatrix}\frac{\partial N}{\partial \xi}\\ \frac{\partial N}{\partial \eta}\end{Bmatrix}$$

$$\begin{Bmatrix}\frac{\partial N}{\partial x}\\ \frac{\partial N}{\partial y}\end{Bmatrix}=\frac{1}{|J|}\begin{bmatrix}\frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi}\\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi}\end{bmatrix}\begin{Bmatrix}\frac{\partial N}{\partial \xi}\\ \frac{\partial N}{\partial \eta}\end{Bmatrix}$$

$$\frac{\partial N}{\partial x}=\frac{1}{|J|}\left[\frac{\partial y}{\partial \eta}\frac{\partial N}{\partial \xi}-\frac{\partial y}{\partial \xi}\frac{\partial N}{\partial \eta}\right]$$

$$\frac{\partial N}{\partial y}=\frac{1}{|J|}\left[\frac{\partial x}{\partial \xi}\frac{\partial N}{\partial \eta}-\frac{\partial x}{\partial \eta}\frac{\partial N}{\partial \xi}\right]$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} & 0 \\ 0 & \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} \\ \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} & \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\{\varepsilon\} = [D'] \begin{Bmatrix} u \\ v \end{Bmatrix} = [D'][N][d]$$

$$\{\varepsilon\} = [D'][N][d]$$

$$\{\varepsilon\} = [B][d]$$

$$[D'] = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} & 0 \\ 0 & \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} \\ \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} & \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} \end{bmatrix}$$

$$\underset{(3\times 8)}{[B]} = \underset{(3\times 2)}{[D']} \underset{(2\times 8)}{[N]}$$

$$[k] = \iint_A [B]^T [D][B]\, t\, dx\, dy$$

$$\iint_A f(x,y)\, dx\, dy = \iint_A f(\xi,\eta)\, |J|\, d\xi\, d\eta$$

$$[k] = \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B]\, t\, |J|\, d\xi\, d\eta$$

The shape function are:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \quad , \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) \quad , \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Their derivatives:

$$\frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta) \quad , \frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1-\eta) \quad , \frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1+\eta) \quad , \frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(1+\eta)$$

and

$$\frac{\partial N_1}{\partial \eta} = -\frac{1}{4}(1-\xi) \quad , \frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(1+\xi) \quad , \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi) \quad , \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi)$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i \quad , \quad J_{12} = \frac{\partial y}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i \quad , \quad J_{22} = \frac{\partial y}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i$$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} N_{1,\xi} & N_{2,\xi} & N_{3,\xi} & N_{4,\xi} \\ N_{1,\eta} & N_{2,\eta} & N_{3,\eta} & N_{4,\eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

$$J_{22} = \frac{1}{4}\left[y_1(\xi-1) + y_2(-1-\xi) + y_3(1+\xi) + y_4(1-\xi)\right]$$

$$J_{12} = \frac{1}{4}\left[y_1(\eta-1) + y_2(1-\eta) + y_3(1+\eta) + y_4(-1-\eta)\right]$$

$$J_{11} = \frac{1}{4}\left[x_1(\eta-1) + x_2(1-\eta) + x_3(1+\eta) + x_4(-1-\eta)\right]$$

$$J_{21} = \frac{1}{4}\left[x_1(\xi-1) + x_2(-1-\xi) + x_3(1+\xi) + x_4(1-\xi)\right]$$

➢ Derive the Element Stiffness Matrix and Equations

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$|J| = J_{11}\,J_{22} - J_{12}\,J_{21}$$

Explicit formulation for |J| for 4 node Element

$$|J| = \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix} = \frac{1}{8}\{x_1 \quad x_2 \quad x_3 \quad x_4\} \begin{bmatrix} 0 & 1-\eta & \eta-\xi & \xi-1 \\ \eta-1 & 0 & \xi+1 & -\xi-\eta \\ \xi-\eta & -\xi-1 & 0 & \eta+1 \\ 1-\xi & \xi+\eta & -\eta-1 & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

➢ Derive the Element Stiffness Matrix and Equations

$$[B] = [D'][N]$$

$$[B] = \frac{1}{|J|} \begin{bmatrix} J_{22}\dfrac{\partial()}{\partial\xi} - J_{12}\dfrac{\partial()}{\partial\eta} & 0 \\ 0 & J_{11}\dfrac{\partial()}{\partial\eta} - J_{21}\dfrac{\partial()}{\partial\xi} \\ J_{11}\dfrac{\partial()}{\partial\eta} - J_{21}\dfrac{\partial()}{\partial\xi} & J_{22}\dfrac{\partial()}{\partial\xi} - J_{12}\dfrac{\partial()}{\partial\eta} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$[B] = \frac{1}{|J|}[B_1 \quad B_2 \quad B_3 \quad B_4]$$

$$[B_i] = \begin{bmatrix} J_{22}\,N_{i,\xi} - J_{12}\,N_{i,\eta} & 0 \\ 0 & J_{11}\,N_{i,\eta} - J_{21}\,N_{i,\xi} \\ J_{11}\,N_{i,\eta} - J_{21}\,N_{i,\xi} & J_{22}\,N_{i,\xi} - J_{12}\,N_{i,\eta} \end{bmatrix}$$

➢ Derive the Element Stiffness Matrix and Equations

$$[B_i]=\begin{bmatrix} J_{22}\,N_{i,\xi}-J_{12}\,N_{i,\eta} & 0 \\ 0 & J_{11}\,N_{i,\eta}-J_{21}\,N_{i,\xi} \\ J_{11}\,N_{i,\eta}-J_{21}\,N_{i,\xi} & J_{22}\,N_{i,\xi}-J_{12}\,N_{i,\eta} \end{bmatrix}$$

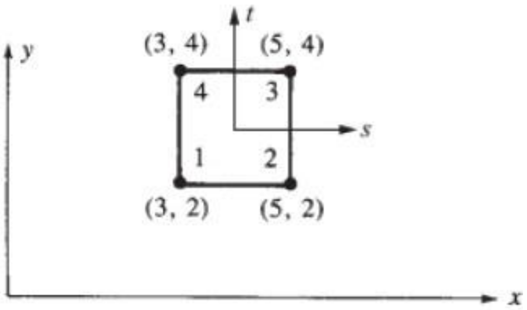
$$J_{22}=\frac{1}{4}\left[y_1(\xi-1)+y_2(-1-\xi)+y_3(1+\xi)+y_4(1-\xi)\right]$$

$$J_{12}=\frac{1}{4}\left[y_1(\eta-1)+y_2(1-\eta)+y_3(1+\eta)+y_4(-1-\eta)\right]$$

$$J_{11}=\frac{1}{4}\left[x_1(\eta-1)+x_2(1-\eta)+x_3(1+\eta)+x_4(-1-\eta)\right]$$

$$J_{21}=\frac{1}{4}\left[x_1(\xi-1)+x_2(-1-\xi)+x_3(1+\xi)+x_4(1-\xi)\right]$$

Evaluate the stiffness matrix for the quadrilateral element shown in Figure using the four-point Gaussian quadrature rule.
Let E = 30×10^6 psi, ν = 0.25 and h=1 in.

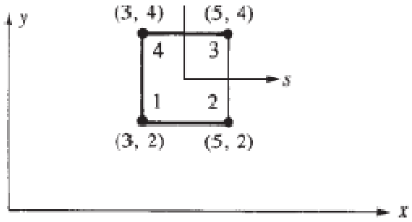


Solution

we evaluate the ***k*** matrix. Using the four-point rule, the four points are:

$$(\xi_1,\eta_1)=(-0.5773,-0.5773)$$
$$(\xi_2,\eta_2)=(-0.5773,0.5773)$$
$$(\xi_3,\eta_3)=(0.5773,-0.5773)$$
$$(\xi_4,\eta_4)=(0.5773,0.5773)$$

$$W_1=W_2=W_3=W_4=1.0$$



$$[k] = [B(-0.5773,-0.5773)]^T[D][B(-0.5773,-0.5773)]$$
$$|[J(-0.5773,-0.5773)]|(1)(1.000)(1.000)$$
$$+[B(-0.5773,0.5773)]^T[D][B(-0.5773,0.5773)]$$
$$|[J(-0.5773,0.5773)]|(1)(1.000)(1.000)$$
$$+[B(0.5773,-0.5773)]^T[D][B(0.5773,-0.5773)]$$
$$|[J(0.5773,-0.5773)]|(1)(1.000)(1.000)$$
$$+[B(0.5773,0.5773)]^T[D][B(0.5773,0.5773)]$$
$$|[J(0.5773,0.5773)]|(1)(1.000)(1.000)$$

$$|[J(-0.5773,-0.5773)]| = \frac{1}{8}[3 \quad 5 \quad 5 \quad 3]$$
$$\begin{bmatrix} 0 & 1-(-0.5773) & -0.5773-(-0.5773) & -0.5773-1 \\ -0.5773-1 & 0 & -0.5773+1 & -0.5773-(-0.5773) \\ -0.5773-(-0.5773) & -(-0.5773)-1 & 0 & -0.5773+1 \\ 1-(-0.5773) & -0.5773+(-0.5773) & -0.5773-1 & 0 \end{bmatrix}$$
$$\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix} = 1.000$$

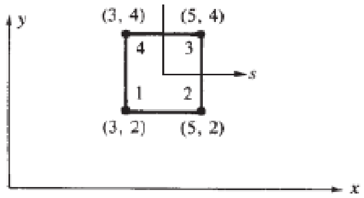
Similarly, $|[J(-0.5773,0.5773)]| = 1.000$

$|[J(0.5773,-0.5773)]| = 1.000$

$|[J(0.5773,0.5773)]| = 1.000$

Even though $|[J]| = 1$ in this example, in general, $|[J]| \neq 1$ and varies in space.

$$[B(-0.5773,-0.5773)] = \frac{1}{|[J(-0.5773,-0.5773)]|}[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]]$$



$$[B_1] = \begin{bmatrix} J_{22}N_{1,\xi} - J_{12}N_{1,\eta} & 0 \\ 0 & J_{11}N_{1,\eta} - J_{21}N_{1,\xi} \\ J_{11}N_{1,\eta} - J_{21}N_{1,\xi} & J_{22}N_{1,\xi} - J_{12}N_{1,\eta} \end{bmatrix}$$

$$J_{22} = \frac{1}{4}\left[y_1(\xi-1) + y_2(-1-\xi) + y_3(1+\xi) + y_4(1-\xi)\right]$$

$$J_{22} = \frac{1}{4}\left[2(-0.5773-1) + 2(-1+0.5773) + 4(1-0.5773) + 4(1+0.5773)\right] = 1.0$$

with similar computations used to obtain J_{12}, J_{11} and J_{21}. Also,

$$N_{1,\xi} = -\frac{1}{4}(1-\eta) = -\frac{1}{4}(1+0.5773) = -0.3943$$

$$N_{1,\eta} = -\frac{1}{4}(1-\xi) = -\frac{1}{4}(1+0.5773) = -0.3943$$

Similarly, $[B_2]$, $[B_3]$, and $[B_4]$ must be evaluated like $[B_1]$, at $(-0.5773, -0.5773)$. We then repeat the calculations to evaluate $[B]$ at the other Gauss points [Eq. (10.4.4a)].

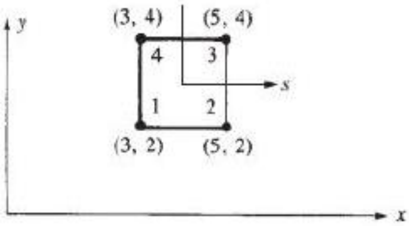
Using a computer program written specifically to evaluate $[B]$ at each Gauss point and then $[k]$, we obtain the final form of $[B(-0.5773, -0.5773)]$ as

$$[B(-0.5773, -0.5773)] =$$

$$\begin{bmatrix} -0.1057 & 0 & 0.1057 & 0 & 0 & -0.1057 & 0 & -0.3943 \\ -0.1057 & -0.1057 & -0.3943 & 0.1057 & 0.3943 & 0 & -0.3943 & 0 \\ 0 & 0.3943 & 0 & 0.1057 & 0.3943 & 0.3943 & 0.1057 & -0.3943 \end{bmatrix} \tag{10.4.4h}$$

with similar expressions for $[B(-0.5773, 0.5773)]$, and so on.

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12 \end{bmatrix} \quad 10^6 \text{ psi}$$



$$\begin{aligned} [k] = {} & [B(-0.5773, -0.5773)]^T[D][B(-0.5773, -0.5773)] \\ & |[J(-0.5773, -0.5773)]|(1)(1.000)(1.000) \\ & + [B(-0.5773, 0.5773)]^T[D][B(-0.5773, 0.5773)] \\ & |[J(-0.5773, 0.5773)]|(1)(1.000)(1.000) \\ & + [B(0.5773, -0.5773)]^T[D][B(0.5773, -0.5773)] \\ & |[J(0.5773, -0.5773)]|(1)(1.000)(1.000) \\ & + [B(0.5773, 0.5773)]^T[D][B(0.5773, 0.5773)] \\ & |[J(0.5773, 0.5773)]|(1)(1.000)(1.000) \end{aligned}$$

$$[k] = 10^4 \begin{bmatrix} 1466 & 500 & -866 & -99 & -733 & -500 & 133 & 99 \\ 500 & 1466 & 99 & 133 & -500 & -733 & -99 & -866 \\ -866 & 99 & 1466 & -500 & 133 & -99 & -733 & 500 \\ -99 & 133 & -500 & 1466 & 99 & -866 & 500 & -733 \\ -733 & -500 & 133 & 99 & 1466 & 500 & -866 & -99 \\ -500 & -733 & -99 & -866 & 500 & 1466 & 99 & 133 \\ 133 & -99 & -733 & 500 & -866 & 99 & 1466 & -500 \\ 99 & -866 & 500 & -733 & -99 & 133 & -500 & 1466 \end{bmatrix}$$

For the rectangular element shown previous Example, assume plane stress conditions

Let E = 30×10^6 psi, ν = 0.3 and displacements:

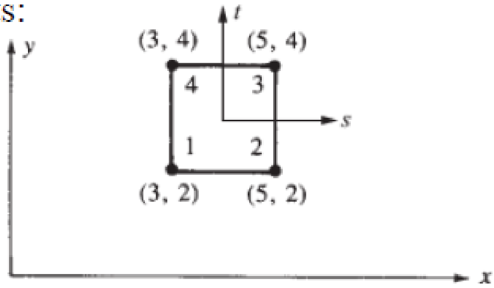
$u_1 = 0$, $v_1 = 0$

$u_2 = 0.001$, $v_2 = 0.0015$

$u_3 = 0.003$, $v_3 = 0.0016$

$u_4 = 0$, $v_4 = 0$

Evaluate the stresses at $s=0$, $t=0$



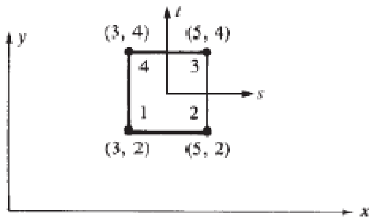
Solution

$$[B] = \frac{1}{|[J]|}[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]]$$

$$[B(0,0)] = \frac{1}{|[J(0,0)]|}[B_1(0,0)] \quad [B_2(0,0)] \quad [B_3(0,0)] \quad [B_4(0,0)]$$

Example 3

$$|[J(0,0)]| = \frac{1}{8}[3 \quad 5 \quad 5 \quad 3]\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix}$$



$$= \frac{1}{8}[-2 \quad -2 \quad 2 \quad 2]\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix}$$

$$|[J(0,0)]| = 1$$

$$[B_i] = \begin{bmatrix} J_{22}\, N_{i,\xi} - J_{12}\, N_{i,\eta} & 0 \\ 0 & J_{11}\, N_{i,\eta} - J_{21}\, N_{i,\xi} \\ J_{11}\, N_{i,\eta} - J_{21}\, N_{i,\xi} & J_{22}\, N_{i,\xi} - J_{12}\, N_{i,\eta} \end{bmatrix}$$

$$J_{22} = \frac{1}{4}\left[y_1(\xi - 1) + y_2(-1-\xi) + y_3(1+\xi) + y_4(1-\xi)\right]$$

$$J_{22} = \frac{1}{4}\left[2(0-1) + 2(-1-0) + 4(1+0) + 4(1-0)\right] = 1$$

Similiarly $J_{12} = 0, \quad J_{11} = 1, \quad J_{21} = 0$

$N_{1,\xi} = -\frac{1}{4}(1-\eta) = -\frac{1}{4}(1-0) = -\frac{1}{4}$ Similarly $N_{2,\xi} = \frac{1}{4}$, $N_{3,\xi} = \frac{1}{4}$ and $N_{4,\xi} = -\frac{1}{4}$

$N_{1,\eta} = -\frac{1}{4}(1-\xi) = -\frac{1}{4}(1-0) = -\frac{1}{4}$ Similarly $N_{2,\eta} = -\frac{1}{4}$, $N_{3,\eta} = \frac{1}{4}$ and $N_{4,\eta} = \frac{1}{4}$

$$[B_1] = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \qquad [B_2] = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad [B_3] = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad [B_4] = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$\{\sigma\} = [D][B]\{d\} =$$

$$= (30)\frac{10^6 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}}{1 - 0.09} \begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0.25 & 0 & -0.25 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.25 & 0 & 0.25 \\ -0.25 & -0.25 & -0.25 & 0.25 & 0.25 & 0.25 & 0.25 & -0.25 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.001 \\ 0.0015 \\ 0.003 \\ 0.0016 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 3.321 \quad 10^4 \\ 1.071 \quad 10^4 \\ 1.471 \quad 10^4 \end{Bmatrix} \text{psi}$$

Higher-Order Shape Functions

- In general, higher-order element shape functions can be developed by adding additional nodes to the sides of the linear element.
- These elements result in higher-order strain variations within each element, and convergence to the exact solution thus occurs at a faster rate using fewer elements.
- Another advantage of the use of higher-order elements is that curved boundaries of irregularly shaped bodies can be approximated more closely than by the use of simple straight-sided linear elements.

Shape function of a quadratic isoparametric element

$$N_1 = \tfrac{1}{4}(1-s)(1-t)(-s-t-1)$$

$$N_2 = \tfrac{1}{4}(1+s)(1-t)(s-t-1)$$

$$N_3 = \tfrac{1}{4}(1+s)(1+t)(s+t-1)$$

$$N_4 = \tfrac{1}{4}(1-s)(1+t)(-s+t-1)$$

or, in compact index notation, we express

$$N_i = \tfrac{1}{4}(1+ss_i)(1+tt_i)(ss_i+tt_i-1)$$

where ***i*** is the number of the shape function

$$s_i = -1, 1, 1, -1 \qquad (i = 1, 2, 3, 4)$$

$$t_i = -1, -1, 1, 1 \qquad (i = 1, 2, 3, 4)$$

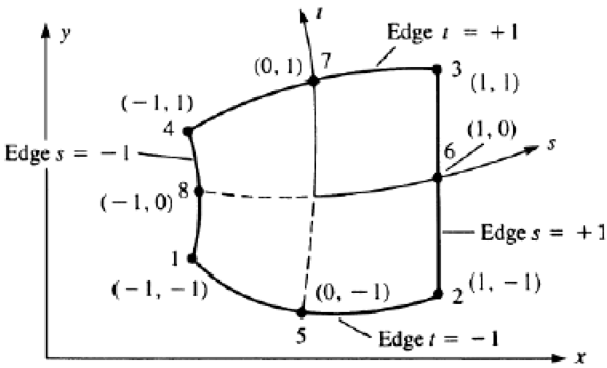


Figure 10–16 Quadratic isoparametric element

For the midside nodes $(i = 5, 6, 7, 8)$,

$$N_5 = \tfrac{1}{2}(1-t)(1+s)(1-s)$$

$$N_6 = \tfrac{1}{2}(1+s)(1+t)(1-t)$$

$$N_7 = \tfrac{1}{2}(1+t)(1+s)(1-s)$$

$$N_8 = \tfrac{1}{2}(1-s)(1+t)(1-t)$$

Shape function of a cubic isoparametric element

For the corner nodes $(i = 1, 2, 3, 4)$,

$$N_i = \tfrac{1}{32}(1+ss_i)(1+tt_i)[9(s^2+t^2)-10]$$

For the nodes on sides $s = \pm 1$ $(i = 7, 8, 11, 12)$,

$$N_i = \tfrac{9}{32}(1+ss_i)(1+9tt_i)(1-t^2)$$

with $s_i = \pm 1$ and $t_i = \pm\frac{1}{3}$.

For the nodes on sides $t = \pm 1$ $(i = 5, 6, 9, 10)$,

$$N_i = \tfrac{9}{32}(1+tt_i)(1+9ss_i)(1-s^2)$$

with $t_i = \pm 1$ and $s_i = \pm\frac{1}{3}$.

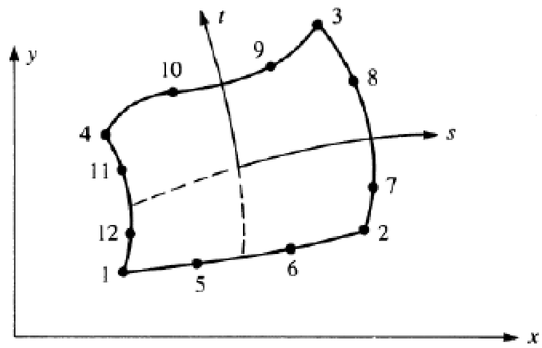


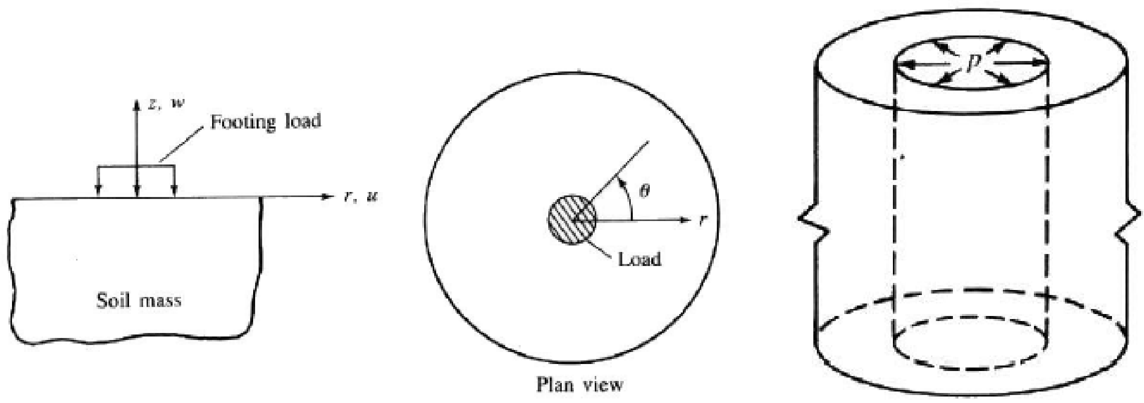
Figure 10–18 Cubic isoparametric element

Definition of an axisymmetric solid

- An axisymmetric solid (or a thick-walled body) of revolution is defined as a 3-D body that is generated by rotating a plane and is most easily described in cylindrical coordinates. Where z is called the axis of symmetry.
- If the geometry, support conditions, loads, and material properties are all axially symmetric (all are independent of θ), then the problem can be idealized as a two-dimensional one.

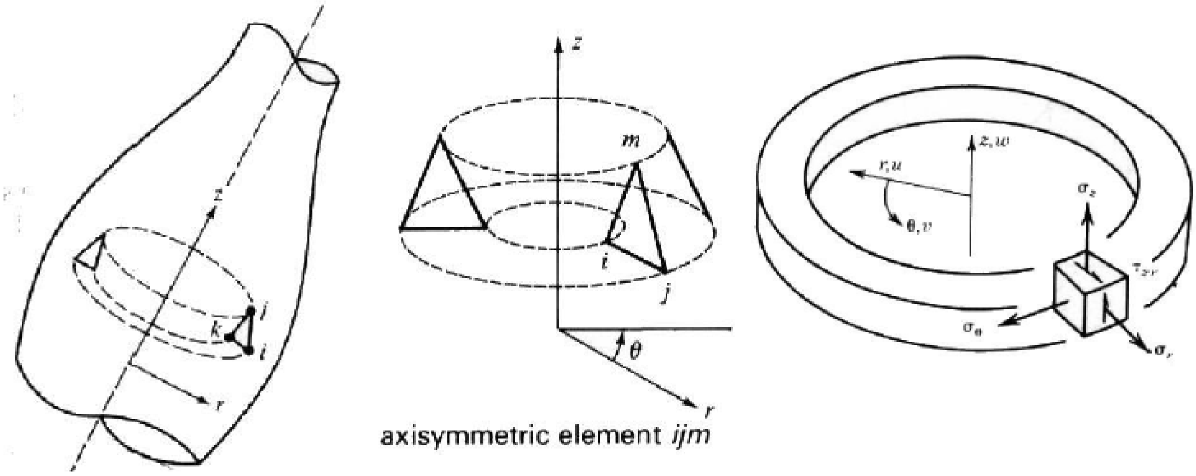
Examples of an axisymmetric solid

Problems such as soil masses subjected to circular footing loads, thick-walled pressure vessels, and a rocket nozzle subjected to thermal and pressure loading can often be analyzed using axisymmetric elements.



FE axisymmetric elements

axisymmetric problems can be analyzed by a finite element of revolution, called axisymmetric elements. Each element consists of a solid ring, the cross-section of which is the shape of the particular element chosen (triangular, rectangular, or quadrilateral elements).
An axisymmetric element has nodal circles rather than nodal points



Equations of Equilibrium:

The three-dimensional elasticity equations in cylindrical coordinates can be summarized as follows

$$\left.\begin{aligned}
&\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + X_r = 0 \\
&\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + Y_\theta = 0 \\
&\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} + Z_b = 0
\end{aligned}\right\}$$

The three-dimensional strain-displacement relationships of elasticity in cylindrical coordinates were ***u, v, w*** are the displacements in the ***r, θ,dz***, respectively, are:

$$
\left.
\begin{aligned}
\varepsilon_r &= \frac{\partial u}{\partial r} &, \quad \gamma_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \\
\varepsilon_\theta &= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} &, \quad \gamma_{rz} &= \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \\
\varepsilon_z &= \frac{\partial w}{\partial z} &, \quad \gamma_{\theta z} &= \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}
\end{aligned}
\right\}
$$

The three-dimensional stress-strain relationships for isotropic elasticity are:

$$
\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{r\theta} \\ \tau_{rz} \\ \tau_{\theta z} \end{Bmatrix}
= \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\
0 & 0 & 0 & & \frac{1-2\nu}{2} & 0 \\
0 & 0 & 0 & 0 & & \frac{1-2\nu}{2}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{r\theta} \\ \gamma_{rz} \\ \gamma_{\theta z} \end{Bmatrix}
$$

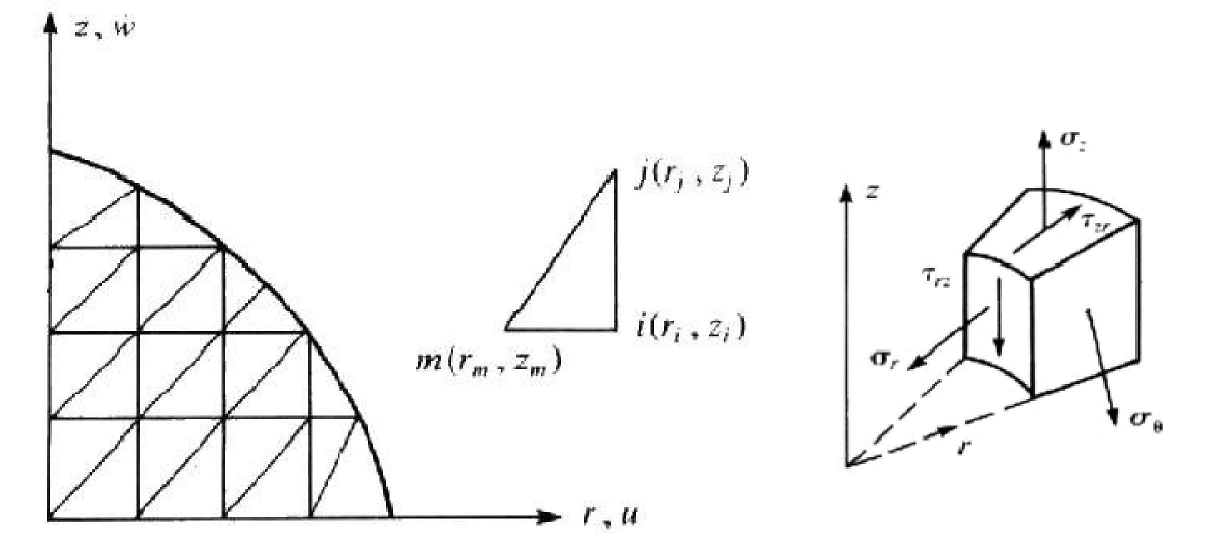
In axisymmetric problems, because of the symmetry about the z -axis, the stresses are independent of the θ coordinate. Therefore, all derivatives with respect to θ vanish and the circumferential (tangent to θ direction) displacement component is zero; therefore,

$$\gamma_{r\theta} = \gamma_{\theta z} = 0 \quad and \quad \tau_{r\theta} = \tau_{\theta z} = 0$$

$$\varepsilon_r = \frac{\partial u}{\partial r} \quad , \quad \varepsilon_\theta = \frac{u}{r} \quad , \quad \varepsilon_z = \frac{\partial w}{\partial z} \quad , \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

Derivation of the Stiffness Matrix and Equations

Step 1: Discretize and Select Element Type



Typical slice through an axisymmetric solid Discretized into triangular elements

$$\{d\} = \begin{Bmatrix} \underline{d}_i \\ \underline{d}_j \\ \underline{d}_m \end{Bmatrix} = \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

(u_i, w_i) → displacement components of ***node i*** in the *r* and *z* directions, respectively.

Step 2: Select Displacement Functions

$$u(r,z) = a_1 + a_2 r + a_3 z$$
$$w(r,z) = a_4 + a_5 r + a_6 z$$

$$\{\psi\} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{Bmatrix} a_1 + a_2 r + a_3 z \\ a_4 + a_5 r + a_6 z \end{Bmatrix} = \begin{bmatrix} 1 & r & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r & z \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$u_i = a_1 + a_2 r_i + a_3 z_i$$
$$u_j = a_1 + a_2 r_j + a_3 z_j$$
$$u_m = a_1 + a_2 r_m + a_3 z_m$$
$$w_i = a_4 + a_5 r_i + a_6 z_i$$
$$w_j = a_4 + a_5 r_j + a_6 z_j$$
$$w_m = a_4 + a_5 r_m + a_6 z_m$$

In Matrix Form

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix}^{-1} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} \qquad \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix}^{-1} \begin{Bmatrix} w_i \\ w_j \\ w_m \end{Bmatrix}$$

Solving for the *a*'s

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} \qquad \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} w_i \\ w_j \\ w_m \end{Bmatrix}$$

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

$$2A = x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j)$$
$$= \alpha_i + \alpha_j + \alpha_m$$

A is the area of the triangle

$$\alpha_i = r_j z_m - z_j r_m \qquad \alpha_j = r_m z_i - z_m r_i \qquad \alpha_m = r_i z_j - z_i r_j$$

$$\beta_i = z_j - z_m \qquad \beta_j = z_m - z_i \qquad \beta_m = z_i - z_j$$

$$\gamma_i = r_m - r_j \qquad \gamma_j = r_i - r_m \qquad \gamma_m = r_j - r_i$$

$$\{u\} = \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\{u\} = \frac{1}{2A} \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

$$\{u\} = \frac{1}{2A} \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{Bmatrix} \alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m \end{Bmatrix}$$

$$u(r,z) = \frac{1}{2A} \left\{ (\alpha_i + \beta_i r + \gamma_i z) u_i + (\alpha_j + \beta_j r + \gamma_j z) u_j + (\alpha_m + \beta_m r + \gamma_m z) u_m \right\}$$

$$w(r,z) = \frac{1}{2A} \left\{ (\alpha_i + \beta_i r + \gamma_i z) w_i + (\alpha_j + \beta_j r + \gamma_j z) w_j + (\alpha_m + \beta_m r + \gamma_m z) w_m \right\}$$

$$\{\psi\} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

$$\{\psi\} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

$$N_i = \frac{1}{2A}\left(\alpha_i + \beta_i r + \gamma_i z\right)$$

$$N_j = \frac{1}{2A}\left(\alpha_j + \beta_j r + \gamma_j z\right)$$

$$N_m = \frac{1}{2A}\left(\alpha_m + \beta_m r + \gamma_m z\right)$$

$$\{\psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

$$\{\psi\} = [N]\{d\}$$

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

Step 3: Define the Strain/Displacement and Stress/Strain Relationships

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{u}{r} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} a_2 \\ a_6 \\ \dfrac{a_1}{r} + a_2 + \dfrac{a_3 z}{r} \\ a_3 + a_5 \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r} & 0 & \frac{\alpha_j}{r} + \beta_j + \frac{\gamma_j z}{r} & 0 & \frac{\alpha_m}{r} + \beta_m + \frac{\gamma_m z}{r} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{bmatrix} B_i & B_j & B_m \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix} \qquad B_i = \begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r} & 0 \\ \gamma_i & \beta_i \end{bmatrix}$$

$$\{\varepsilon\} = [B]\{d\}$$

B is a function of ***r*** and ***z***

Stress Strain Relationship

$$\{\sigma\} = [D][B]\{d\}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$

Step 4 :Derive the Element Stiffness Matrix and Equations

$$[k] = \iiint_V [B]^T [D][B]\, dV$$

$$[k] = 2\pi \iint_A [B]^T [D][B]\, r\, dr\, dz$$

1) Numerical integration (Gaussian quadrature)
2) Explicit multiplication and term-by-term integration.
3) Evaluate [B] at a centroidal point of the element

$$r = \bar{r} = \frac{r_i + r_j + r_m}{3} \qquad z = \bar{z} = \frac{z_i + z_j + z_m}{3}$$

$$[B(\bar{r}, \bar{z})] = [\bar{B}]$$

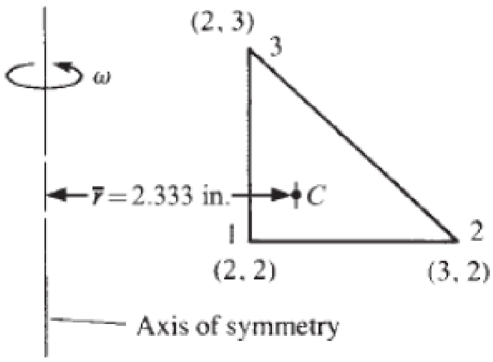
$$[k] = 2\pi \bar{r} A [\bar{B}]^T [D][\bar{B}]$$

Example 1

For the element of an axisymmetric body rotating with a constant angular velocity ω ***= 100 rev/min***.
Evaluate the approximate body force matrix, include the weight of the material, where the weight density ρ_w **is 0.283** ***lb/in³***.
The coordinates of the element (in inches) are shown in the figure.

The body forces per unit volume evaluated at the centroid of the element are

Z_b =0.283 lb/in³



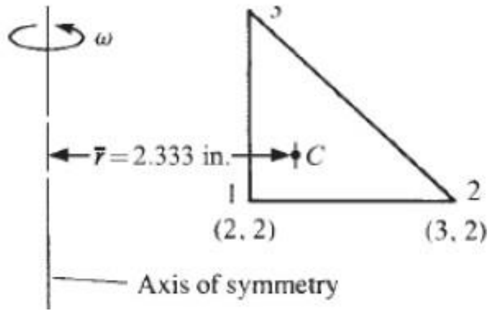
$$\overline{R}_b = \omega^2 \rho \bar{r} = \left[100 \frac{\text{rev}}{\text{min}} \left(2\pi \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right)\right]^2 \frac{(0.283 \text{ lb/in}^3)}{(32.2 \times 12) \text{ in./s}^2}(2.333 \text{ in.})$$

$$\overline{R}_b = 0.187 \text{ lb/in}^3$$

$$\frac{2\pi\bar{r}A}{3} = \frac{2\pi(2.333)(0.5)}{3} = 2.44 \text{ in}^3$$

$$f_{b1r} = (2.44)(0.187) = 0.457 \text{ lb}$$

$$f_{b1z} = -(2.44)(0.283) = -0.691 \text{ lb} \qquad \text{(downward)}$$



All r-directed and z-directed nodal body forces are equal

$$f_{b2r} = 0.457 \text{ lb} \qquad f_{b2z} = -0.691 \text{ lb}$$

$$f_{b3r} = 0.457 \text{ lb} \qquad f_{b3z} = -0.691 \text{ lb}$$

UNIT 5

DYNAMIC ANALYSIS

Syllabus

Dynamic Analysis: Formulation of finite element model, element matrices, evaluation of Eigen values and Eigen vectors for a stepped bar and a beam.

OBJECTIVE:

To learn the application of FEM equations for dynamic analysis

OUTCOME:

Solve dynamic problems where the effect of mass matters during the analysis

UNIT-V
DYNAMIC ANALYSIS

Dynamics is a special branch of mechanics where inertia of accelerating masses must be considered in the force-deflection relationships. **In** order to describe motion of the mass system, a component with distributed mass is approximated by a finite number of mass points. Knowledge of certain principles of dynamics is essential to the formulation of these equations.

Every structure is associated with certain frequencies and mode shapes of free vibration (without continuous application of load), based on the distribution of mass and stiffness in the structure. Any time-dependent external load acting on the structure, whose frequency matches with the natural frequencies of the structure, causes resonance and produces large displacements leading to failure of the structure. Calculation of natural frequencies and mode shapes is there for every important.

In general, for a system with on' degrees of freedom, stiffness 'k' and mass 'm' are represented by stiffness matrix [K] and mass matrix [M] respectively.

Then

$$([K] - \omega^2 [M])\{u\} = \{0\}$$

$$([M]^{-1}[K] - \omega^2 [I])\{u\} = \{0\}$$

Here, [M] is the mass matrix of the entire structure and is of the same order, say n x n, as the stiffness matrix [K]. This is also obtained by assembling element mass matrices in a manner exactly identical to assembling element stiffness matrices. The mass matrix is obtained by two different approaches, as explained subsequently.

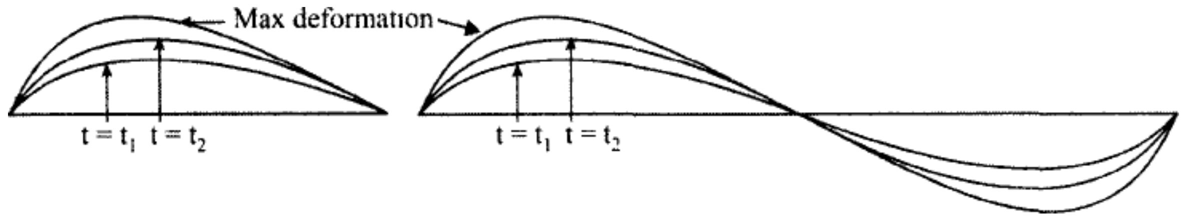
A structure with 'n' DOF will therefore have 'n' eigen values and 'n' eigenvectors. Some eigen values may be repeated and some eigen values maybe complex, in pairs. The equation can be represented in the standard form,

$[A]\{x\}_i = \lambda_i\{x\}_i$.

In dynamic analysis, ω_i, indicates ith natural frequency and $\{X\}_i$ indicates ith natural mode of vibration.

A natural mode is a *qualitative* plot of nodal displacements. In every natural mode of vibration, all the points on the component will reach their maximum values at the same time and will pass through zero displacements at the same time. Thus, in a particular mode, all the points of a component will vibrate with the same frequency and their relative displacements are indicated by

the components of the corresponding eigen vector. These relative (or proportional) displacements at different points
on structure remain same at every time instant for undamped free vibration.



Hence, without loss of generality, {u(t)} can be written as {u}.

Since {u} = {O} forms a trivial solution, the homogeneous system of equations

([A] - λ[I]) {u} = {O}

gives a non-trivial solution only when

([A]-λ[I])= {O},

which implies

Det([A] - λ[I]) = o.

This expression, called *characteristic equation,* results in nth order polynomial in A, and will therefore have n roots. For each λ, the corresponding eigenvector {u}. can be obtained from the n homogeneous equations represented by

([K] - λ[M]) {u} = {O}.

The mode shape represented by {u(t)}gives relatives values of displacements in various degrees of freedom.

NORMALIZATION

The equation of motion of free vibrations ([K] $-\omega^2$[M]) {u} = {O} is a system of homogeneous equations (right side vector zero) and hence does not give unique numerical solution.

Mode shape is a set of relative displacements in various degrees of freedom, while the structure is vibrating in a particular frequency and is usually expressed in normalized form, by following one of the
three normalization methods explained here.

(a) The maximum value of anyone component of the eigenvector is equated to 'I' and, so, all other components will have a value less than or equal to ' 1' .

(b) The length of the vector is equated to '1 ' and values of all components are divided by the length of this vector so that each component will have a value less than or equal to '1'.

(c) The eigenvectors are usually normalized so that

$$\{u\}_i^T [M] \{u\}_i = 1 \quad \text{and} \quad \{u\}_i^T [K] \{u\}_i = \lambda_i$$

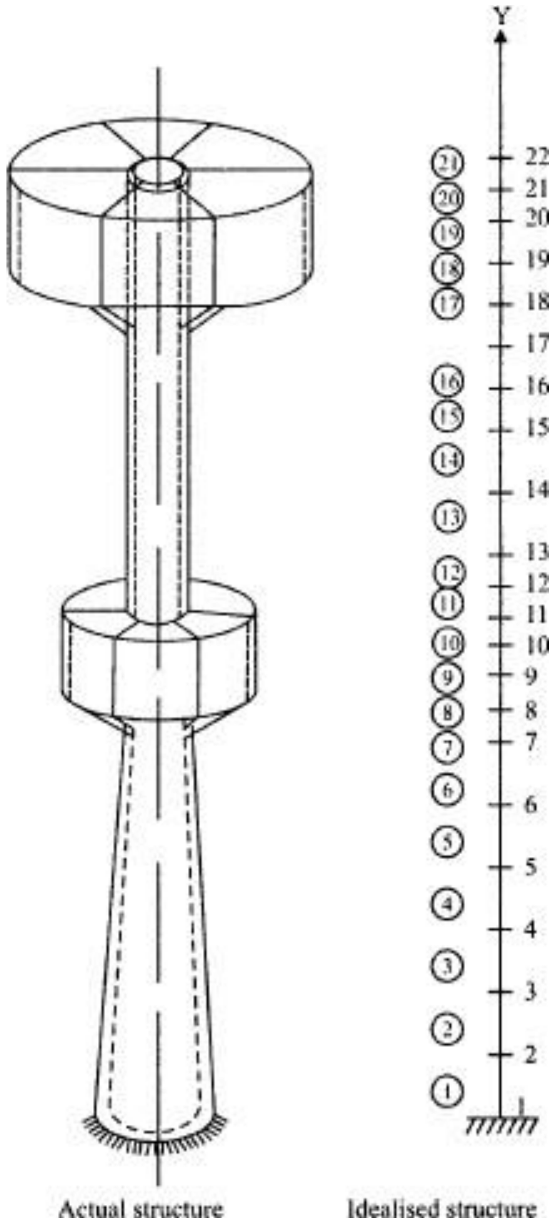
For a positive definite symmetric stiffness matrix of size n x n, the Eigen values are all real and eigenvectors are orthogonal

i.e.,

$$\{u\}_i^T [M] \{u\}_j = 0 \quad \text{and} \quad \{u\}_i^T [K] \{u\}_j = 0 \quad \forall \; i \neq j$$

MODELLING FOR DYNAMIC ANALYSIS

Solution for any dynamic analysis is an iterative process and, hence, is time -consuming. Geometric model of the structure for dynamic analysis can be significantly simplified, giving higher priority for proper representation of distributed mass. An example of a simplified model of a water storage tank is shown in Fig. Below, representing the central hollow shaft by long beam elements and watertanks at two levels by a few lumped masses and short beam elements of larger moment of inertia.



MASS MATRIX

Mass matrix [M] differs from the stiffness matrix in many ways:

(i) The mass of each element is equally distributed at all the nodes of that element

(ii) Mass, being a scalar quantity, has same effect along the three translational degrees of freedom (u, v and w) and is not shared

(iii) Mass, being a scalar quantity, is not influenced by the local or globa coordinate system. Hence, no transformation matrix is used for converting mass matrix from element (or local) coordinate system to structural (or global) coordinate system.

Two different approaches of evaluating mass matrix [M] are commonly considered.

(a) Lumped mass matrix

Total mass of the element is assumed equally distributed at all the nodes of the element in each of the translational degrees of freedom. Lumped mass is not used for rotational degrees of freedom. Off-diagonal elements of this matrix are all zero. This assumption *excludes dynamic coupling* that exists between different nodal displacements.

Lumped mass matrices [M] of some elements are given here.

Lumped mass matrix of truss element with 1 translational DOF per node along its local X-axis

$$[M]=\frac{\rho AL}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Lumped mass matrix of plane truss element in a 2-D plane with2 translational DOF per node (Displacements along X and Y coordinate axes)

$$[M]=\frac{\rho AL}{2}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Please note that the same lumped mass is considered in each translational degree of freedom (without proportional sharing of mass between them) at each node.

Lumped mass matrix of a beam element in X-V plane, with its axis along x-axis and with two DOF per node (deflection along Y axis and slope about Z axis) is given below. Lumped mass is not considered in the rotational degrees of freedom.

$$[M]=\frac{\rho AL}{2}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Note that lumped mass terms are not included in 2nd and 4th rows, as well as columns corresponding to rotational degrees of freedom.

Lumped mass matrix of a CST element with 2 DOF per node. In this case, irrespective of the shape of the element, mass is assumed equally distributed at the three nodes. It is distributed equally in all DOF at each node, without any sharing of mass between different DOF

$$[M]= \frac{\rho AL}{3}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Consistent mass matrix

Element mass matrix is calculated here, *consistent* with the assumed displacement field or element stiffness matrix. [M] is a banded matrix of the same order as the stiffness matrix. This is evaluated using the same

interpolating functions which are used for approximating displacement field over the element. It yields more accurate results but with more computational cost. Consistent mass matrices of some elements are given here.

Consistent mass matrix of a Truss element along its axis (in local coordinate system)

$$\{u\}^T = [u \quad v]$$

$$[N]^T = [N_1 \quad N_2]$$

where, $N_1 = \frac{(1-\xi)}{2}$

and $N_2 = \frac{(1+\xi)}{2}$

$$[M]= \int_v [N]\rho[N]^T \, dV = \int_0^L A[N]\rho[N]^T$$

$$dx = \int_{-1}^{+1} A\,\rho\,[N][N]^T\,(\det J)(dx/d\xi)d\xi$$

Here, $x = N_1x_1 + N_2x_2 = \frac{(x_1+x_2)}{2} + \frac{(x_2-x_1)\xi}{2}$

and $dx = \frac{dx}{d\xi}.d\xi = \det J\, d\xi = \left(\frac{L}{2}\right)d\xi$

Using the values of integration in natural coordinate system,

$$[M] = \rho A\left(\frac{L}{2}\right)\int_{-1}^{+1}\begin{bmatrix}(1-\xi)/2 \\ (1+\xi)/2\end{bmatrix}\left[(1-\xi)/2 \quad (1+\xi)/2\right]d\xi$$

$$= \frac{\rho AL}{8}\begin{bmatrix}\int(1-\xi)^2 d\xi & \int(1-\xi^2)d\xi \\ \int(1-\xi^2)d\xi & \int(1+\xi)^2 d\xi\end{bmatrix}$$

$$= \frac{\rho AL}{8}\begin{bmatrix}(\xi-\xi^2+\xi^3/3) & (\xi-\xi^3/3) \\ (\xi-\xi^3/3) & (\xi+\xi^2+\xi^3/3)\end{bmatrix}$$

$$= \frac{\rho AL}{8}\begin{bmatrix}8/3 & 4/3 \\ 4/3 & 8/3\end{bmatrix} = \frac{\rho AL}{6}\begin{bmatrix}2 & 1 \\ 1 & 2\end{bmatrix}$$

Consistent mass *matrix (if a* **Plane** *Truss element,* inclined to global X-axis -Same elements of I-D mass matrix are repeated in two dimensions (along X and Y directions) without sharing mass between them. Mass terms in X and Y directions are uncoupled.

$$[M] = \frac{\rho AL}{6}\begin{bmatrix}2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2\end{bmatrix}$$

Consistent mass matrix of a Space *Truss element,* inclined to X-Y plane) -Same elements of 1-0 mass matrix are repeated in three dimensions (along X, Y and Z directions) without sharing mass between them.

$$[M] = \frac{\rho AL}{6}\begin{bmatrix}2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2\end{bmatrix}$$

Consistent *mass matrix of a Beam element*

$[M]=\rho A\left(\frac{L}{2}\right)\int\{H\}^T\{H\}d\xi$ with Hermite shape functions $\{H\}$ as used in a beam element.

$$=\frac{\rho AL}{128}\int\begin{bmatrix}2(2-3\xi+\xi^3)\\ L(1-\xi+\xi^2+\xi^3)\\ 2(2+3\xi-\xi^3)\\ L(-1-\xi+\xi^2+\xi^3)\end{bmatrix}\times$$

$$\left[2(2-3\xi+\xi^3)\quad L(1-\xi-\xi^2+\xi^3)\quad 2(2+3\xi-\xi^3)\quad L(-1-\xi+\xi^2+\xi^3)\right]d\xi$$

$$=\frac{\rho AL}{420}\begin{bmatrix}156 & 22L & 54 & -13L\\ 22L & 4L^2 & 13L & -3L^2\\ 54 & 13L & 156 & -22L\\ -13L & -3L^2 & -22L & 4L^2\end{bmatrix}$$

Consistent mass matrix of a CST element in a 2-D plane

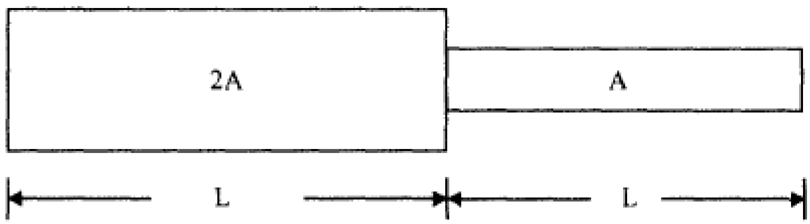
$$[N]^T=\begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3\end{bmatrix}$$

$$[M]=\int[N]\rho[N]^T\,dV=t\int[N]\rho[N]^T\,dA$$

$$=\frac{\rho tA}{12}\begin{bmatrix}2 & 0 & 1 & 0 & 1 & 0\\ & 2 & 0 & 1 & 0 & 1\\ & & 2 & 0 & 1 & 0\\ & & & 2 & 0 & 1\\ & \text{Sym} & & & 2 & 0\\ & & & & & 2\end{bmatrix}$$

Note: Natural frequencies obtained using lumped mass matrix are LOWER than exact values.

Example 1 : Find the natural frequencies of longitudinal vibrations of the unconstrained stepped shaft of areas A and 2A and of equal lengths (L), as shown below.



Solution: Let the finite element model of the shaft be represented by 3 nodes and 2 truss elements (as only longitudinal vibrations are being considered) as shown below.

1. Longitudinal vibration of bar

Finite element equation,

$$\{[K]-[m]\omega^2\}\{u\} = \{F\}$$

where, Stiffness matrix, $[K] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Mass matrix, $[m] = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ for consistent mass matrix

$[m] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for lumped mass matrix

2. Transverse vibration of beam

Finite element equation,

$$\{[K]-[m]\omega^2\}\{u\} = \{F\}$$

where, Stiffness matrix, $[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$

Mass matrix, $[m] = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$

for consistent mass matrix

$[m] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ for lumped mass matrix

Example ***Find the natural frequency of longitudinal vibration of the unconstrained stepped bar as shown***

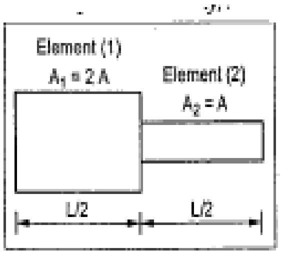


Fig. (i).

Given:

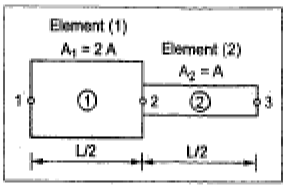


Fig. (ii).

Element (1)	**Element (2)**
Area, $A_1 = 2A$	Area, $A_2 = A$
Length, $L_1 = \frac{L}{2}$	Length, $L_2 = \frac{L}{2}$
Young's modulus, $E_1 = E$	Young's modulus, $E_2 = E$
Density, $\rho_1 = \rho$	Density, $\rho_2 = \rho$

To find: Natural frequencies of the rod.

☺***Solution:*** The bar with two element and 3 nodes are as shown in Fig.(*ii*). The stiffness matrix of the two elements are,

Stiffness matrix for Element (1),

$$[K_1] = \frac{A_1 E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2AE}{\frac{L}{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{4AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_1] = \frac{2AE}{L}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix} \quad (\text{columns } \mathbf{1}, \mathbf{2})$$

Similarly, Stiffness matrix for Element (2),

$$[K_2] = \frac{A_2 E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{AE}{\frac{L}{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_2] = \frac{2AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{matrix} \mathbf{2} \\ \mathbf{3} \end{matrix} \quad (\text{columns } \mathbf{2}, \mathbf{3}) \qquad \ldots(2)$$

Assemble the stiffness matrix,

$$[K] = \frac{2AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix} \quad (\text{columns } \mathbf{1}, \mathbf{2}, \mathbf{3}) \qquad \ldots(3)$$

Mass Matrix for Element (1),

$$[m_1] = \frac{\rho_1 A_1 L_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[m_1] = \frac{\rho \times 2A \times \frac{L}{2}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[m_1] = \frac{\rho A L}{12}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{matrix}1\\2\end{matrix} \quad \text{(columns 1, 2)} \qquad \dots(4)$$

Similarly, Mass matrix for Element (2),

$$[m_2] = \frac{\rho_2 A_2 L_2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho A \times \left(\frac{L}{2}\right)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[m_2] = \frac{\rho A L}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{matrix}2\\3\end{matrix} \quad \text{(columns 2, 3)} \qquad \dots(5)$$

Assemble the mass matrix,

$$[m] = \frac{\rho A L}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{matrix}1\\2\\3\end{matrix} \quad \text{(columns 1, 2, 3)} \qquad \dots(6)$$

Since, the bar is unconstrained (no degrees of freedom is fixed), the finite element equation is

$$\{[K] - [m]\omega^2\}\{u\} = \{P\}$$

Substitute [K] and [m] values

$$\left[\frac{2AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

Applying boundary conditions,

$$P_1 = P_2 = P_3 = 0$$

[No degrees of freedom is fixed]

We set the determinant of the coefficient matrix equal to zero, we have

$$\left|\frac{2AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0 \qquad \dots(7)$$

Divide both sides by $\left(\frac{2AE}{L}\right)$,

$$\left|\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\omega^2 \frac{\rho A L}{12}}{\frac{2AE}{L}}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0$$

Divide both sides by $\left(\frac{2AE}{L}\right)$,

$$\left|\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\omega^2 \frac{\rho A L}{12}}{\frac{2AE}{L}}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\rho L^2 \omega^2}{24E}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0 \qquad \dots(8)$$

Take, $\beta^2 = \frac{\rho L^2 \omega^2}{24E}$

Equation (8) can be rewritten as,

$$\left|\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \beta^2\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix} 2(1-2\beta^2) & -2(1+\beta^2) & 0 \\ -2(1+\beta^2) & 3(1-2\beta^2) & -(1+\beta^2) \\ 0 & -(1+\beta^2) & (1-2\beta^2) \end{vmatrix} = 0$$

$$\Rightarrow 2(1-2\beta^2)[3(1-2\beta^2)^2 - (1+\beta^2)^2] + 2(1+\beta^2)[-2(1+\beta^2)(1-2\beta^2)] = 0$$

By simplifying the above equation, we get

$$\Rightarrow 18\,\beta^2 (1 - 2\,\beta^2)(\beta^2 - 2) = 0 \qquad \dots (9)$$

The roots of equation (9) give the natural frequencies of the bar.

We know that, $\beta^2 = \dfrac{\rho\, L^2\, \omega^2}{24\, E}$

when, $\beta^2 = 0 \Rightarrow \omega_1^2 = 0 \Rightarrow \omega_1 = 0$

when, $\beta^2 = \dfrac{1}{2} \Rightarrow \omega_2^2 = \dfrac{12\,E}{\rho\, L^2} \Rightarrow \omega_2 = 3.46\left[\dfrac{E}{(\rho\, L^2)}\right]^{\frac{1}{2}}$ rad/s

when, $\beta^2 = 2 \Rightarrow \omega_3^2 = \dfrac{48\,E}{\rho\, L^2} \Rightarrow \omega_3 = 6.92\left[\dfrac{E}{(\rho\, L^2)}\right]^{\frac{1}{2}}$ rad/s

∴ Natural frequencies are,

$$\omega_1 = 0$$

$$\omega_2 = 3.46\left[\frac{E}{(\rho\, L^2)}\right]^{\frac{1}{2}}$$

$$\omega_3 = 6.96\left[\frac{E}{(\rho\, L^2)}\right]^{\frac{1}{2}}$$

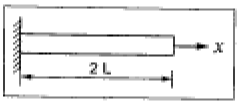
Result: Natural frequencies of longitudinal vibration,

$$\omega_1 = 0$$

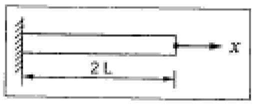
$$\omega_2 = 3.46\left[\frac{E}{(\rho\, L^2)}\right]^{\frac{1}{2}} \text{ rad/s}$$

$$\omega_3 = 6.96\left[\frac{E}{(\rho\, L^2)}\right]^{\frac{1}{2}} \text{ rad/s}$$

For the bar as shown in Fig.(i) with length 2 L, modulus of elasticity E, mass density ρ, and cross sectional area A, determine the first two natural frequencies.



Given:



Length, L = 2 L

Young's modulus, E = E

Mass density, ρ = ρ

Cross-sectional area, A = A

To find: Natural frequencies.

Solution: We can divide the bar into two elements as shown in Fig.(iii).

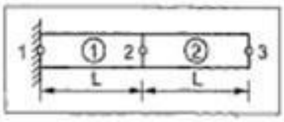


Fig. (iii).

Stiffness matrix for element (1):

$$[K_1] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{matrix} 1 \\ 2 \end{matrix} \qquad (\text{columns } 1,\ 2)$$

Similarly,

Element (2):

$$[K_2] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{matrix} 2 \\ 3 \end{matrix} \qquad (\text{columns } 2,\ 3)$$

Assembling the element matrix,

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \qquad \dots (1)$$

(columns 1, 2, 3)

Lumped mass matrix or consistent mass matrix can be used for solving the problem.

Lumped mass matrix for element (1):

$$[m_1] = \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix} \quad (\text{columns } \mathbf{1}\ \mathbf{2})$$

Similarly,

Element (2):

$$[m_2] = \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{matrix} \mathbf{1} \\ \mathbf{2} \end{matrix} \quad (\text{columns } \mathbf{1}\ \mathbf{2})$$

Assembling the mass matrix,

$$[m] = \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix} \quad (\text{columns } \mathbf{1}\ \mathbf{2}\ \mathbf{3}) \qquad \dots (2)$$

Global matrix, for bar element,

$$\{[K] - \omega^2 [m]\}\{u\} = \{P\}$$

$$\left[\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \qquad \dots (3)$$

Applying the boundary conditions,

$$u_1 = 0 \text{ (fixed)}, \qquad P_1 = 0$$
$$u_2 = u_2 \qquad P_2 = 0$$
$$u_3 = u_3 \qquad P_3 = 0$$

$$\Rightarrow \left[\left[\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\right]\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

In the above equation, $u_1 = 0$, so, neglect first row and first column of [K] and [m] matrix. The final reduced equation is,

$$\left\{\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 0 \qquad \dots (4)$$

In the above equation, $u_1 = 0$, so, neglect first row and first column of [K] and [m] matrix. The final reduced equation is,

$$\left\{\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 0 \qquad \dots (4)$$

To obtain a solution to the set of homogeneous equation in equation (4), we set the determinant of the coefficient matrix equal to zero.

$$\left|\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0 \qquad \dots (5)$$

Divide both sides by ρAL,

$$\left|\frac{AE}{\rho A L^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \frac{\rho A L}{2 \times \rho A L}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0 \qquad [\because \lambda = \omega^2]$$

$$\left|\frac{E}{\rho L^2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\lambda}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

Take, $\mu = \dfrac{E}{\rho L^2}$

$$\left|\mu\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\lambda}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\Rightarrow \quad \begin{vmatrix} (2\mu - \lambda) & -\mu \\ -\mu & \left(\mu - \frac{\lambda}{2}\right) \end{vmatrix} = 0 \qquad \ldots (6)$$

$$\Rightarrow \quad \left[(2\mu - \lambda)\left(\mu - \frac{\lambda}{2}\right)\right] - [\mu^2] = 0$$

$$\Rightarrow \quad \left(2\mu^2 - \mu\lambda - \mu\lambda + \frac{\lambda^2}{2} - \mu^2\right) = 0$$

$$\Rightarrow \quad \mu^2 - 2\mu\lambda + \frac{\lambda^2}{2} = 0$$

$$\Rightarrow \quad \frac{\lambda^2}{2} - 2\mu\lambda + \mu^2 = 0 \qquad \ldots (7)$$

By solving the quadratic equation (7),

$$\lambda = -(-2\mu) \pm \frac{\sqrt{4\mu^2 - \frac{4}{2}\mu^2}}{\left(\frac{2}{2}\right)} \qquad \left[\because \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

$$\boxed{\lambda = 2\mu \pm \mu\sqrt{2}} \qquad \ldots (8)$$

$$\lambda = \mu[2 \pm \sqrt{2}]$$

$$\therefore \quad \lambda_1 = 3.41\,\mu, \quad \lambda_2 = 0.585\,\mu$$

Natural frequencies are,

We know that, $\lambda = \omega^2$

$$\Rightarrow \quad \omega = \sqrt{\lambda}$$

$$\Rightarrow \quad \omega_1 = \sqrt{3.41\,\mu}$$

$$\boxed{\omega_1 = 1.85\sqrt{\mu} \text{ rad/s}}$$

$$\therefore \quad \omega_1 = 1.85\sqrt{\frac{E}{\rho L^2}} \text{ rad/s} \qquad \left[\because \mu = \frac{E}{\rho L^2}\right]$$

Similarly, $\omega_2 = \sqrt{0.585\,\mu}$

$$\boxed{\omega_2 = 0.76\sqrt{\mu} \text{ rad/sec}}$$

$$\omega_2 = 0.76\sqrt{\frac{E}{\rho L^2}} \text{ rad/s}$$

Result: Natural frequencies are,

$$\omega_1 = 1.85\sqrt{\frac{E}{\rho L^2}} \text{ rad/s}$$

$$\omega_2 = 0.76\sqrt{\frac{E}{\rho L^2}} \text{ rad/s}$$

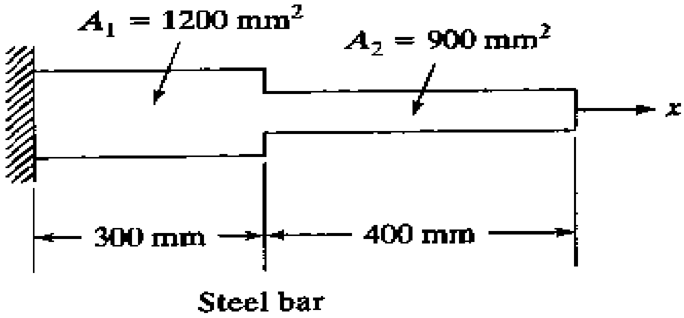
MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

Subject: FINITE ELEMENT METHODS

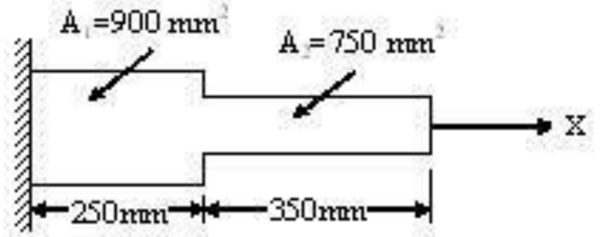
UNIT – V

TUTORIAL - V

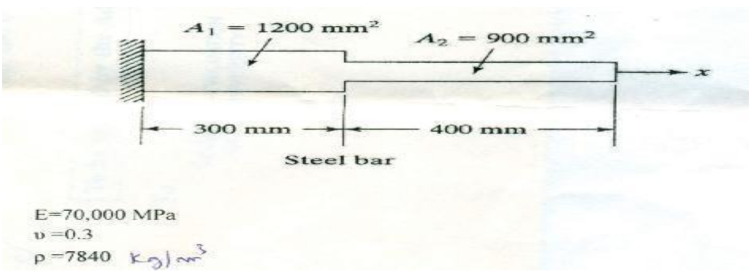
1. Determine natural frequencies for a Steel bar as shown in figure.



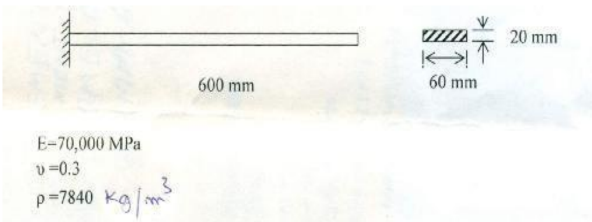
2. a.) Write a short note on damping.
   b.) Consider axial vibration of the steel bar shown in Figure., Develop the global stiffness and mass matrices Determine the natural frequencies and mode shapes using the characteristic polynomial technique.



3. Consider axial vibration of the steel bar shown in Fig. a) Develop the global stiffness and mass matrices b) By hand calculations, determine the lowest natural frequency and mode shape 1and 2



4. Write the step by step procedure to determine the frequencies and nodal displacements of the steel cantilever beam shown in Figure.



5. Explain the Overview of Commercial software's like ANSYS, ABAQUES .

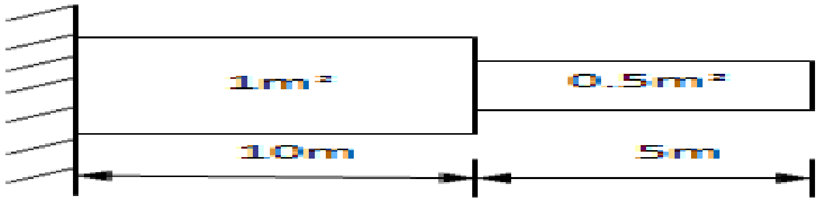
MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

Subject: FINITE ELEMENT METHODS

UNIT – V

ASSIGNMENT - V

1. Determine the Eigen values and Eigen Vectors for the stepped bar as shown in Figure, take density as 7850 kg/m^3 and E= 30X10^6N/m^2?

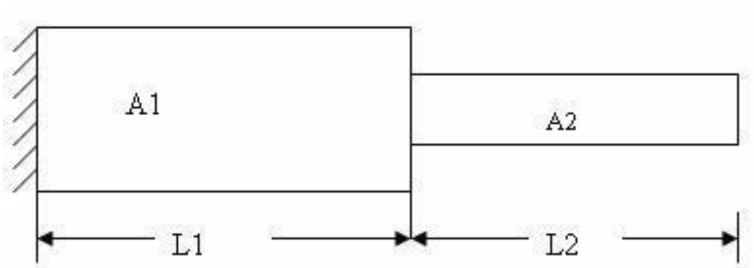


2. Define a.) Eigen value and Eigenvector

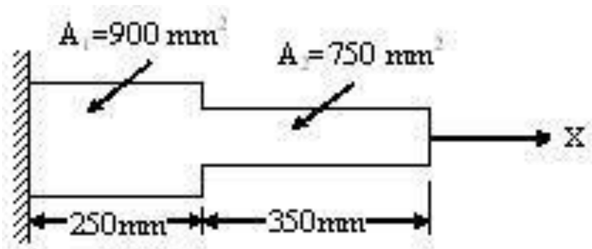
   b.) Dynamic analysis

3. Determine natural frequencies and corresponding mode shapes for the figure

   Take L_1=1m, L_2=2m, A_1=2m^2, A_2=1m^2, ρ = 7850 kg/m^3, E = 200Gpa



4. Consider axial vibration of the steel bar shown in Figure.6,
   i) Develop the global stiffness and mass matrices
   ii) Determine the natural frequencies and mode shapes using the characteristic polynomial technique.



5.) Write short note on a.) Eigen vectors for a stepped beam b.) Evaluation of Eigen values

PREVIOUS QUESTION PAPERS

R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Regular/supplementary Examinations, April/May 2019

Finite Element Methods

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B

Part A is compulsory which carriers 25 marks and Answer all questions.

Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

***

PART-A (25 Marks)

1). a What is meant by finite Element method **[2M]**

b Name the weighted residual techniques? **[3M]**

c Write down the expression of stiffness matrix for a truss element. **[2M]**

d Define plane strain problem. **[3M]**

e What is CST element? **[2M]**

f Write down the shape functions for an axisymmetric triangular element. **[3M]**

g Write the governing equation for a steady flow heat conduction. **[2M]**

h Write down the expression of stiffness matrix for a beam element. **[3M]**

i What is meant by discretization and assembling? **[2M]**

j What is the difference between static and dynamic analysis? **[3M]**

PART-B (50 MARKS)

SECTION-I

2 Describe advantages, disadvantages and applications of finite element analysis. **[10M]**

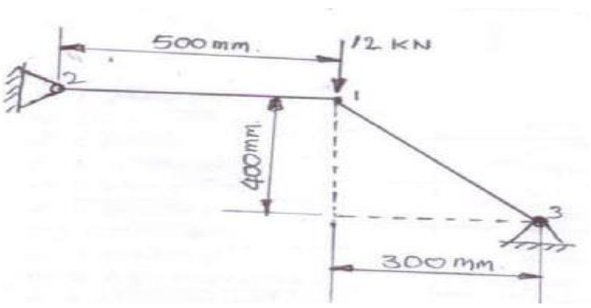
OR

3 The following equation is available for a physical phenomena

$\frac{d^2 y}{dx^2}$ -10x^2=5; 0<x<1, Boundary Conditions; y(0) =0, y(1) =0, Using Galarkin method of weighted residual find an approximate solution of the above differential equation. **[10M]**

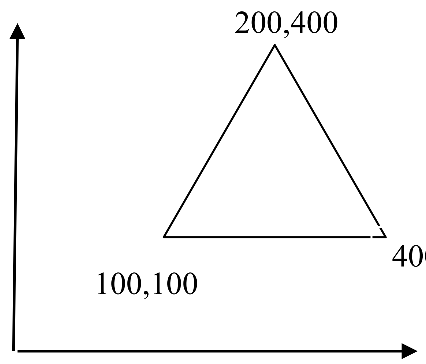
SECTION-II

4 For the two bar truss shown in figure, determine the displacement at node 1 and stresses in element2, Take E=70GPa, A= 200mm^2. **[10M]**



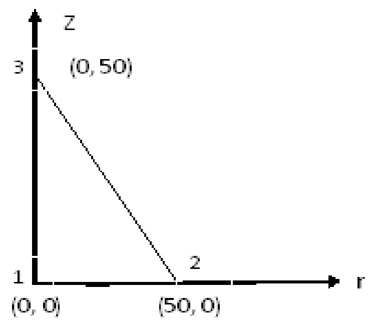
OR

5 For the plane stress element shown in figure the nodal displacements are **[10M]**
U_1= 2.0mm, V_1=1.0mm
U_2= 1.0 mm, V_2= 1.5mm, U_3= 2.5mm, V_3=0.5mm, Take E= 210GPa, ν= 0.25, t=10mm. Determine the strain-Displacement matrix [B].



SECTION-III

6 For axisymmetric element shown in figure, determine the strain-displacement matrix. Let E = 2.1×10^5N/mm^2 and ν= 0.25. The co-ordinates shown in figure are in millimeters. **[10M]**



OR

7 Evaluate the following integral using Gaussian quadrature, so that the result is exact. **[10M]**

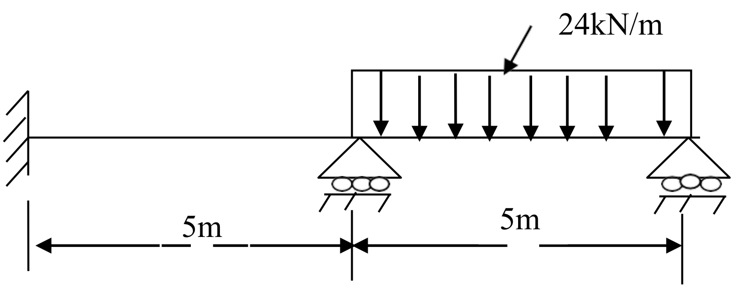
$f(r) = \int_{-1}^{1} (\frac{1}{1+x^2}$ +2x –sinx) dx

SECTION-IV

8 Estimate the temperature distribution in a fin whose cross section is 15mm X 15mm and 500mm long. Take Thermal conductivity as 50W/m-k and convective heat transfer coefficient as 75 W/m^2-k at 25°C. The base temperature is assumedto be constant and its value may be taken as 900°C. And also calculate the heat transfer rate? **[10M]**

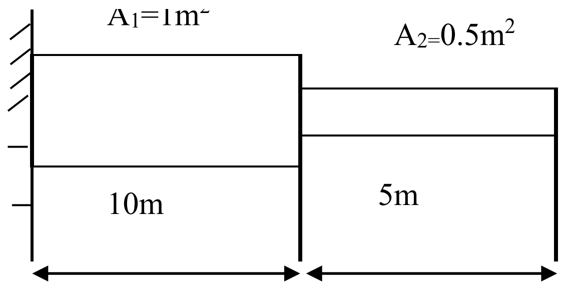
OR

9 For the beam loaded as shown in figure, determine the slope at the simple supports. Take E=200GPa, I=4×10^6m^4. **[10M]**



SECTION-V

10 Determine the Eigen values and Eigen vectors for the beam shown in figure



$E=30x10^5 N/m^2$ **[10M]**

$\rho=0.283 kg/m^3$

OR

11 Write short note on **[10M]**

(a) Eigen vectors for a stepped beam
(b) Evaluation of Eigen values.

****

Code No: **R15A0322**

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Regular Examinations, April/May 2018

Finite Element Method

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B
Part A is compulsory which carriers 25 marks and Answer all questions.
Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

******

PART- A

1a. What is the shape function? Give its practical importance. [2]

b) Briefly discuss the Gherkin's approach in solving FEA problems [3]

c) Define is axisymmetric element with 2 practical applications [2]

d. What are the differences between plane stress and plane strain problems [3]

e. Briefly discuss the advantages of Axisymmetric Elements [2]

f. Describe the shape functions in natural coordinates for 2-D Quadrilateral element. [3]

g. Write the governing equation for a steady flow heat conduction [2]

h. Write short notes on applications of FEM [3]

i. What are the practical importance of Eigen values and Eigen vectors [2]

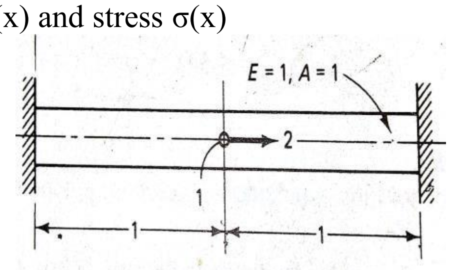
j. Write the Gradient matrix[B] for CST element. [3]

PART – B

10 * 5 = 50 Marks

2. SECTION-1 [5]

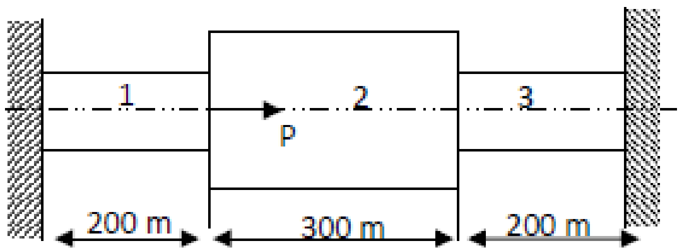
(a) A rod fixed at its ends is subjected to a varying body force as shown in Figure.1. Use the Rayleigh-ritz method with an assumed displacement field $u=a_0+a_1x+a_2x^2$ to determine displacement u(x) and stress σ(x)



(b) Write the Potential function for a continuum under all possible loads and indicate all the variables involved. Also express the total potential of general finite element in terms of nodal displacements [5]

OR

3. An axial load $P = 200X10^3$ N is applied on a bar shown in figure, determine nodal displacements, stress in each material and reaction forces. If A_1= 2400 mm^2, A_2= 600 mm^2, A_3= 2000 mm^2,E_1=70GPa,E_2=200GPa, E_3=67GPa [10]



4. [5]

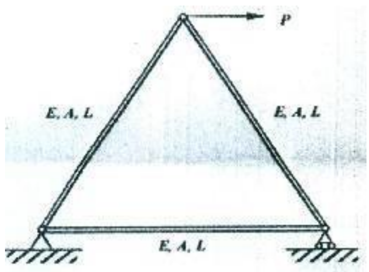
SECTION - II

(a) Derive the B Matrix (relating strains and nodal displacements) for an iso parametric triangular element with linear interpolation for the geometry as well as field variables.

b) Explain why the above element is popularly known as CST. Discuss about the advantages and disadvantages of the element [5]

OR

5. For the truss shown in figure establish the element stiffness matrices and assemble the global stiffness matrix for the active degrees of freedom and determine a) Nodal displacements b) Stress in the members and c) The reaction at the roller support, Take E= 100 Gpa. Area of c/section = 100 mm^2 Length = 100 cm, P = 100 kN. [10]

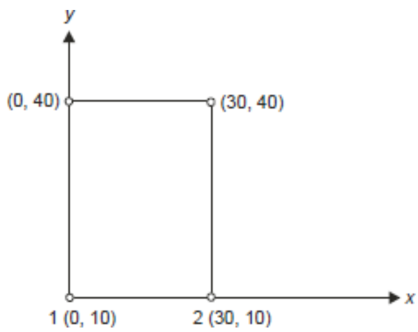


SECTION-III

6. Derive the B Matrix (relating strains and nodal displacements) for an axi-Symmetric iso parametric triangular element with linear interpolation for the geometry as well as field variables. [10]

OR

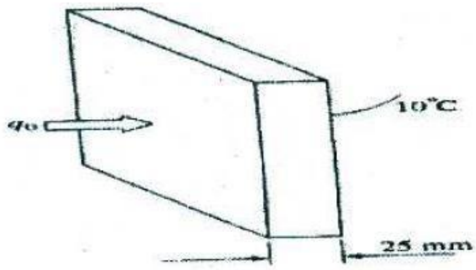
7.(a) Consider a quadrilateral element as shown in figure, Evaluate Jacobian matrix and strain-Displacement matrix at local coordinates ξ =0.5, η = 0.5. [7]



(b) Evaluate the integral $\int_{-1}^{+1} [3e^x + 2x^2 + \frac{1}{(3x+4)}]\, dx$ using one point and two point Gauss quadrature. [3M]

SECTION-IV

8. Heat is entering into a large plate at the rate of q_0=-300w/m2 as shown in Figure, the plate is 25 mm think. The outside surface of the plate is maintained at a temperature of10 ^{0}C. Using two finite elements, solve for the vector of nodal temperatures T, thermal conductivity k=1.0 w/m^{0}c [10]

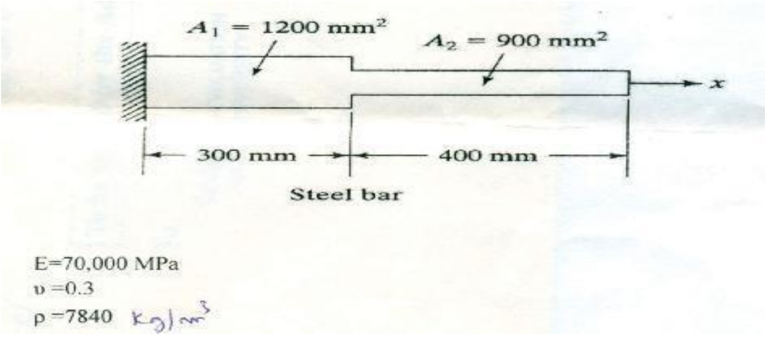


OR

9. Estimate the temperature profile in **a** fin of diameter 25 mm, whose length is 400mm. The thermal conductivity of the fin material is 50 W/m K and heat transfer coefficient over the surface of the fin is 50 W/m^2 K at 30^{0}C. The tip is insulated and the base is exposed to a temperature of 150 ^{0}C. Evaluate the temperatures at points separated by 100 mm each. [10]

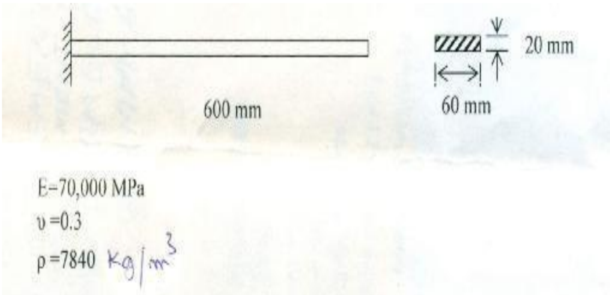
SECTION-V

10. Consider axial vibration of the steel bar shown in Fig. a) Develop the global stiffness and mass matrices b) By hand calculations, determine the lowest natural frequency and mode shape 1and 2 [10]



OR

11. Write the step by step procedure to determine the frequencies and nodal displacements of the steel cantilever beam shown in Fig. [10]



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Code No: **R15A0322** **R15**

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester supplementary Examinations, Nov/Dec 2018

Finite Element Methods

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B

Part A is compulsory which carriers 25 marks and Answer all questions.

Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

PART – A

1.a. Briefly discuss weighted residual method for giving approximate solutions for complicated domains [2M]

b. Write the stiffness matrix for 1-d element with linear interpolation functions [3M]

c. Differentiate iso-parametric, sub-parametric, and super parametric elements? [2M]

d. What is the difference between plane truss and space truss? [3M]

e. What are the uses of natural coordinates in 2d- Quadrilateral elements [2M]

f. What are the suitable applications of axi-symmetric elements in FEM? [3M]

g. Write the governing equation for FEA formulation for a fin [2M]

h Express the stiffness matrix for a 1-D conduction problem [3M]

i. What do you understand by mode shapes? [2M]

j. How principle of minimum potential energy is useful in dynamic analysis of systems [3M]

PART – B 10 * 5 = 50 Marks

SECTION-I

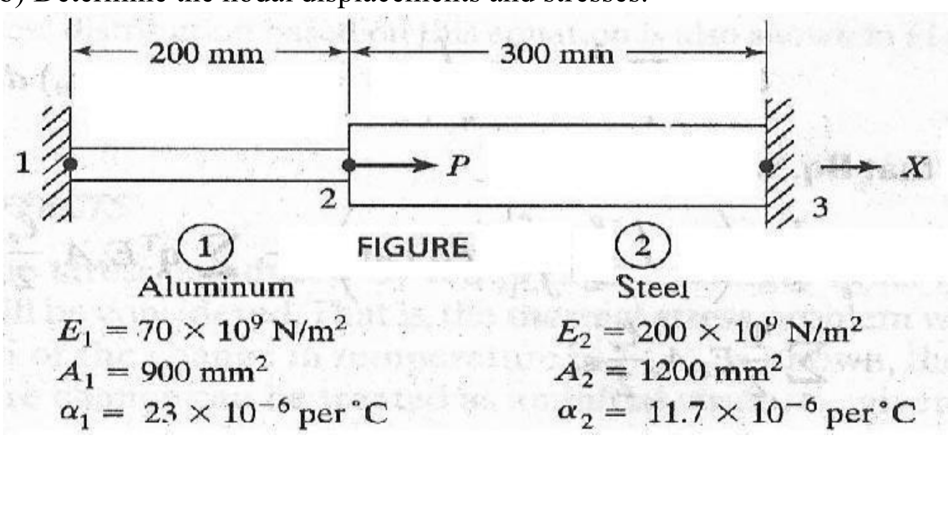
2. Derive the equations equilibriums for 3-D body [10M]

OR

3. An axial load P=300X103N is applied at 200 C to the rod as shown in Figure below. The temperature is the raised to 600 C . [10M]

a) Assemble the K and F matrices.

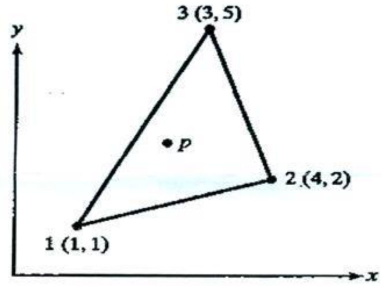
b) Determine the nodal displacements and stresses.



SECTION-II

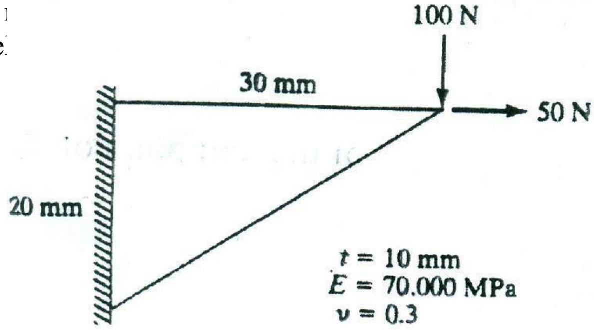
4. a) Write the difference between CST and LST elements [3M]

b) For point P located inside the triangle shown in the figure below the shape functions N1 and N2 are 0.15 and 0.25, respectively. Determine the x and y coordinates of point P. [7M]



OR

5 For the configuration shown in Fig. determine the deflection at the point of load application using a one-element ı used, comment on the stress values in the e [10M]



S

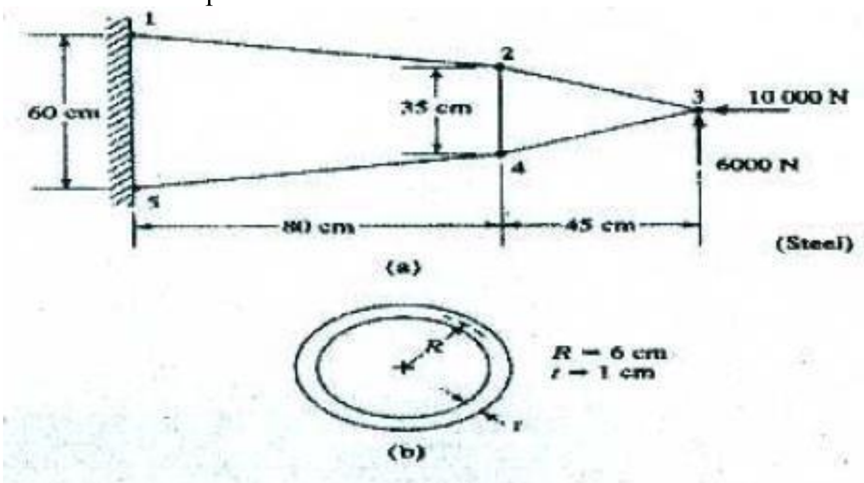
6. Derive the strain displacement matrix for axisymmetric triangular element Discuss advantages of axisymmetric modelling in FEM [10M]

OR

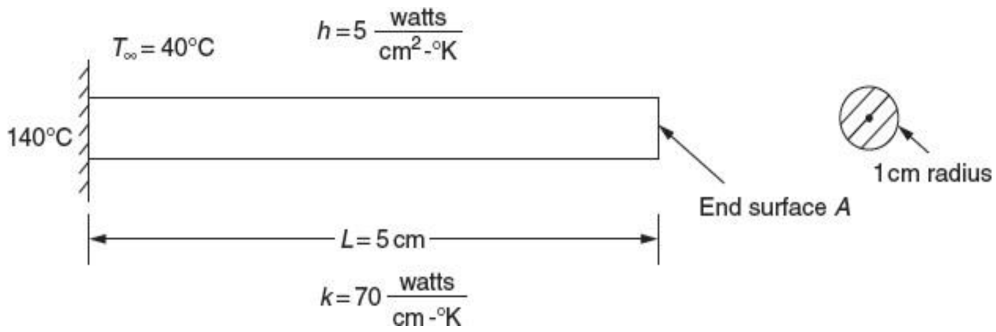
7. Figure shows a five – member steel frame subjected to loads at the free end. The cross section of each member is a tube of wall thickness t=1 cm and mean radius=6cm. Determine the following: [10M]

a) The displacement of node 3 and

b) The maximum axial compressive stress in a member



8. Find the temperature distribution in the one-dimensional fin shown in Figure below using two finite elements. [10M]



OR

9. (a) A 20-cm thick wall of an industrial furnace is constructed using fireclay bricks that have a thermal conductivity of k = 2 W/m-°C. During steady state operation, the furnace wall has a temperature of 800°C on the inside and 300°C on the outside. If oneof the walls of the furnace has a surface area of 2 m^2 (with 20-cm thickness), find the rate of heat transfer and rate of heat loss through the wall. [5M]
(b) A metal pipe of 10-cm outer diameter carrying steam passes through a room. The walls and the air in the room are at a temperature of 20°C while the outer surface of the pipe is at a temperature of 250°C. If the heat transfer coefficient for free convection from the pipe to the air is h = 20 W/m^2-°C find the rate of heat loss from the pipe. [5M]

10. For the two-bar truss shown in Figure below, determine the nodal displacements, element stresses and support reactions. A force of P=1000kN is applied at node-1. Assume E=210GPa and A=600mm2 for each element. [10M]

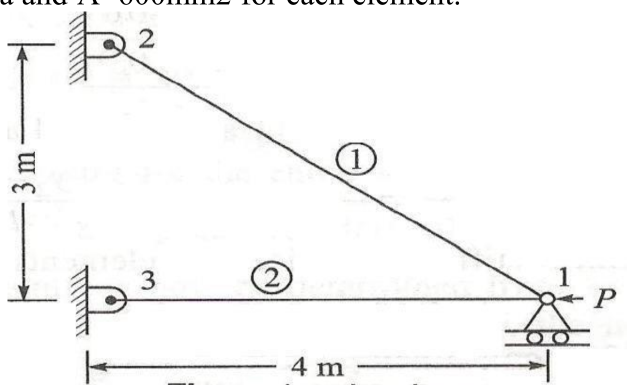


Figure two-bar truss.

OR

11. A bar of length 1 m; cross sectional area 100 mm2; density of 7 gm/cc and Young's modulus 200Gpa is fixed at both the ends. Consider the bar as three bar elements and determine the first two natural frequencies and the corresponding mode shapes. Discuss on the accuracy of the obtained solution [10M]

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MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY
B. Tech III Year II Semester
FINITE ELEMENT METHODS

\- - -

1.a) Derive the interpolation functions at all nodes for the quadratic serendipity element.

b) Evaluate the integral by using one and two-point Gaussian quadrature and compare with exact value.

$$I = \int_{-1}^{+1}\int_{-1}^{+1} (x^3 + x^2 y + xy^2 + \sin 2x + \cos 2y)\,dx\,dy$$

2.a) Clearly explain the finite element formulation for an axisymmetric shell with an axisymmetric loading. Determine the matrix relating strains and nodal displacements for an axisymmetric triangular element.

b) Establish the Hermite shape functions for a beam element Derive the equivalent nodal point loads for a u.d.l. acting on the beam element in the transverse direction and also determine stiffness matrix.

3.a) Write about different boundary considerations in beams.

b) Determine the support reactions and maximum vertical deflection for the continuous beam shown in Figure.1.

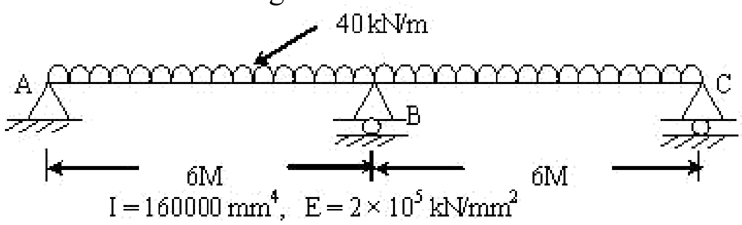

Figure.1

4.a) Discuss in detail about 2D heat conduction in Composite slabs using FEA.

b) Using the isoparametric element, find the Jacobian and inverse of Jacobian matrix for the element shown in Fig.2, 3(a) & 3(b) for the following cases.

i) Determine the coordinate of a point P in x-y coordinate system for the $\xi = 0.4$ and $\eta = 0.6$.

ii) Determine the coordinate of the Q in ξ and η system for the x = 2.5 and y = 1.0.

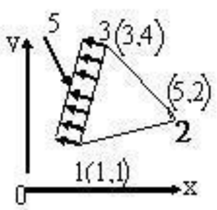

Fig. 2

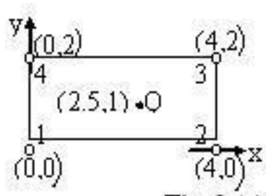

Fig. 3 (a)

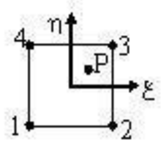

Fig. 3 (b)

5. Calculate the temperature distribution and the heat dissipating capacity of a fin shown in Figre.4. The thermal conductivity of the material is 200 W/ m. K. The surface transfer coefficient is 0.5 W/m^2K. The ambient temperature is 30 °C. the thickness of the fin is 1 cm.

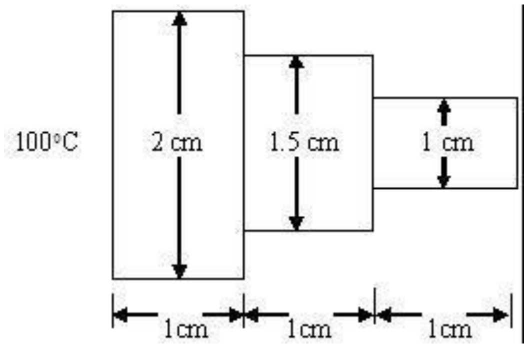

Figure.4

6.a) Write the steps involved in finite-element analysis of a typical problem.

b) Determine the nodal displacements, element stresses and support reactions for the bar as shown in Figure 5. Take E = 200 × 10^9 N/m^2.

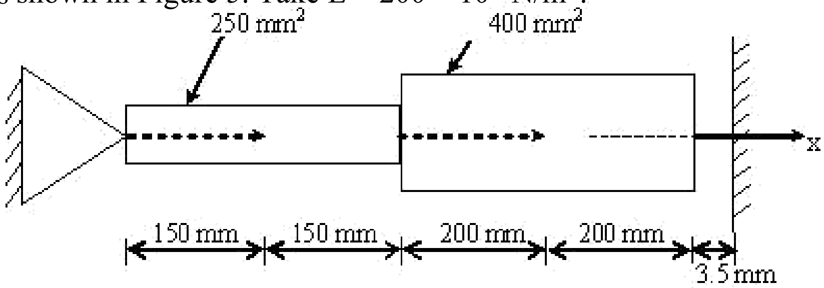

Figure.5

7.a) Derive the equilibrium equation for an elastic continuum using potential energy by displacement approach.

b) Explain the following methods used for the formulation of element characteristics and load matrices:

i) Variational approach ii) Galerkin approach

8.a) With an example differentiate Between Lumped mass, Consistent mass and Hybrid mass matrix and derive for truss element.

b) Consider axial vibration of the steel bar shown in Figure.6,

i) Develop the global stiffness and mass matrices

ii) Determine the natural frequencies and mode shapes using the characteristic polynomial technique.

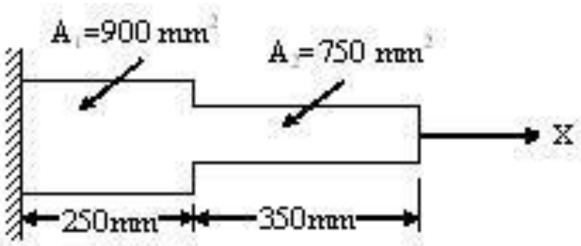

Figure.6

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

B. Tech III Year II Semester
FINITE ELEMENT METHODS
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1)a) Discuss in detail about the concepts of FEM formulation .How is that FEM emerged as powerful tool. Discuss in detail about applications of finite element method.

b)Derive an equation for finding out the potential energy by Rayleigh –Ritz method. Using Rayleigh – Ritz method, find the displacement of the midpoint of the rod shown in Fig.1. Assume E = 1, A = 1, ρ g = 1 by using linear and quadratic shape functions concept.

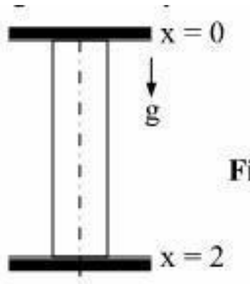

Fig. 1

2. a) Discuss in detail about Linear and Quadratic shape functions with examples.
   b) For the truss shown in fig.2 determine the displacements at point B and stresses in the bars by considering linear and quadratic shape functions.

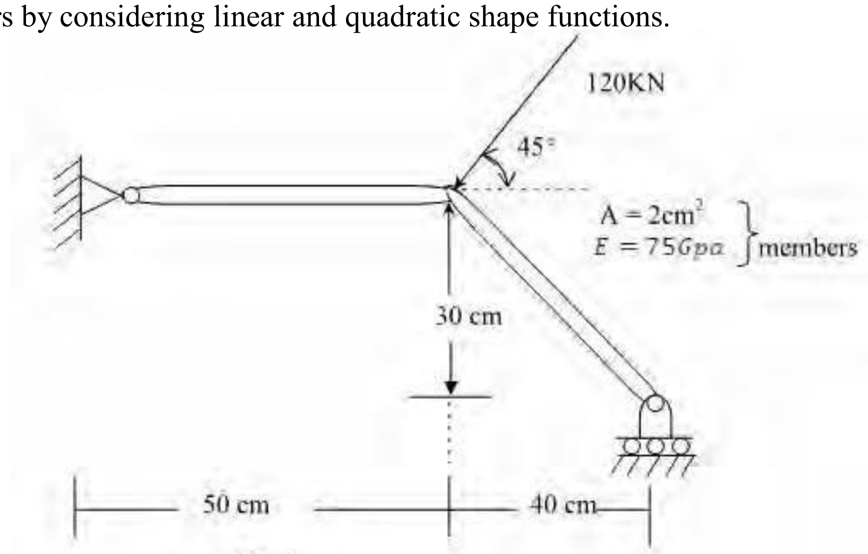

Fig. 2

3. a) Consider axial vibration of the Aluminum bar shown in Fig.3, (i) develop the global stiffness and (ii) determine the nodal displacements and stresses using elimination approach and with help of linear and quadratic shape function concept. Assume Young's Modulus E = 70Gpa.

b) Determine the mass matrix for truss element with an example.

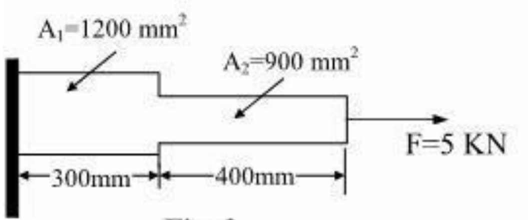

Fig. 3

4. a) Establish the shape functions for a 3 – noded triangular element.

b) Find the deformed configuration, and the maximum stress and minimum stress locations for the rectangular plate loaded as shown in the fig.4. Solve the problem using 2 triangular elements. Assume thickness = 10cm; E = 70 Gpa, and υ = 0.33.

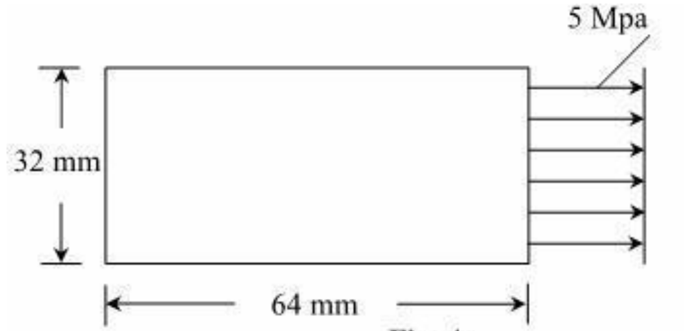

Fig. 4

5. a) Determine the shape functions for 4 – nodded quadrilateral element.

b) For a beam and loading shown in fig.5, determine the slopes at 2 and 3 and the vertical deflection at the midpoint of the distributed load.

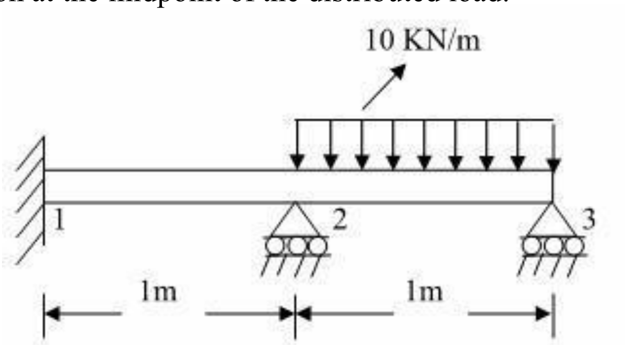

Fig.5

6. a) Clearly explain the finite element formulation for an axisymmetric shell with an axisymmetric loading.Determine the matrix relating strains and nodal displacements for an axisymmetric triangular element.

b) Determine the temperature distribution in a straight fin of circular c/s. Use three one dimensional linear elements and consider the tip is insulated. Diameter of fin is 1 cm, length is 6 cm, h = 0.6 W/cm^2 –C, φ_∞ = 25^0C and base temperature is φ_1 =80^0C.

7. a) Determine the element stresses, strains and support reactions for the given bar problem as shown in Fig. 6

=1.2 mm ; L =150 mm; P = 60000 N ; $E = 2 X 10^4 \frac{N}{m}$; A = 250 mm.

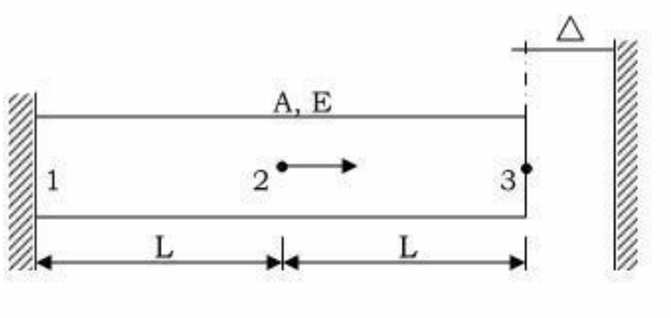

Fig. 6

b) What are shape functions? Indicate briefly the role of shape functions in FEM analysis.

8. a) Derive one dimensional steady state heat conduction equation.

b) An axisymmetric triangular element is subjected to the loading as shown in fig.7 the load is distributed throughout the circumference and normal to the boundary. Derive all the necessary equations and derive the nodal point loads.

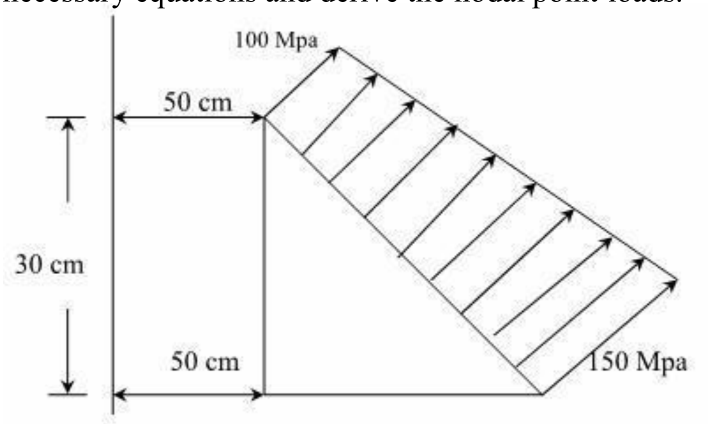

Fig.7

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Finite Element Methods

1.a) Write the strain stress relations based on generalized Hooke's law and derive the elasticity matrix for 3-D field problems.

c) Describe the standard procedure to be followed for understanding the finite element method step by step with suitable example.

2.a) Derive the stiffness matrix of axial bar element with quadratic shape functions based on first principles.

c) Calculate the nodal displacements and forces for the stepped bar with the stiffness values of 10 kN/m and 18 kN/m and a load of 32 kN is subjected at the end of the stepped bar and other end of the bar is fixed.

3.a) Derive the shape functions and stiffness matrix of a two nodded beam element.

c) Derive the load vector for the beam element when a uniformly distributed load is applied.

4.a) For a plane strain problem, the nodal displacements are u_1 = 4.4 μm, u_2 = 2.2 μm, u_3=2.2 μm, v_1 = 3.8 μm, v_2 = 2.9 μm, v_3 = 4.5 μm. Take E=200 GPa, μ = 0.3 and t=10mm. Find the stresses, principal stresses. The coordinates of triangular element are 1(5,25), 2(15,5) and 3(25,15). All dimensions are in millimeters.

c) Show that the stiffness for a triangular element is $[B]^T[D][B]$ At using variational principle. Where A=area of the triangle and t= thickness.

5.a) Compute the strain displacement matrix and also the strains of a axisymmetric triangular element with the coordinates r_1 = 3 cm, z_1 = 4 cm, r_2 = 6 cm, z_2 = 5 cm, r_3= 5 cm, z_3 = 8 cm. The nodal displacement values are u_1 = 0.01 mm, w_1 = 0.01 mm, u_2 = 0.01 mm, w_2 = -0.04 mm, u_3 = -0.03 mm, w_3 = 0.07 mm

b) Differentiate between Axi symmetric elements and symmetric elements with suitable examples.

6.a) Explain the methodology to estimate the stiffness matrix of four noded quadrilateral element.

b) Evaluate $\int [e^{2x} + x^3 + 1 / (x^2 + 2)]\, dx$ over the limits -1 and +1 using one point and three point quadrature formula and compare with exact solution.

7.a) What are different thermal applications of finite element analysis? Compare the structural analysis with thermal analysis.

b) Calculate the temperature distribution in the fin of 10 mm diameter, which is exposed to the convective b.c. of 40 W/m^2 K with 30^0 C. The base of the fin is exposed to a heat flux of 450 kW/m^3 and the thermal conductivity of fin material is 30 W/m K.

8. Determine natural frequencies and corresponding mode shapes for the figure 8.

Take L_1=1m, L_2=2m, A_1=2m^2, A_2=1m^2, ρ = 7850 kg/m^3, E = 200Gpa

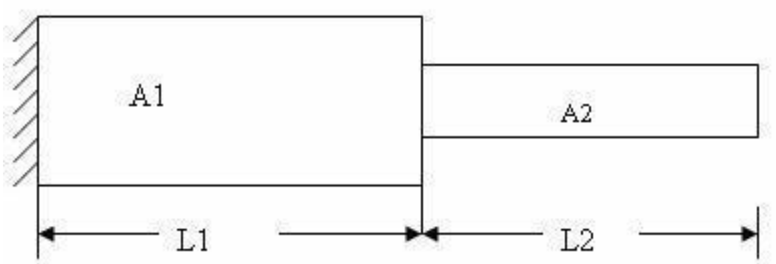


Fig: 8

R15

Code No: **R15A0322**

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Supplementary Examinations, October/November 2020

Finite Element Method

(ME)

Roll No										

Time: 2 hours **Max. Marks: 75**

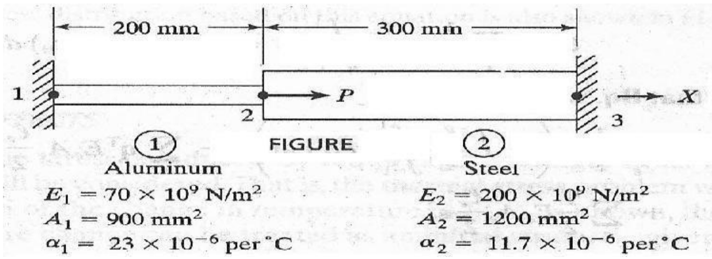
Answer Any **Four** Questions

All Questions carries equal marks.

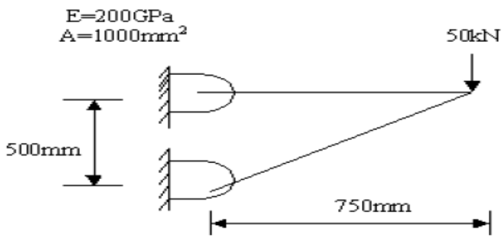
***

1 Briefly describe the general procedure of finite element analysis.

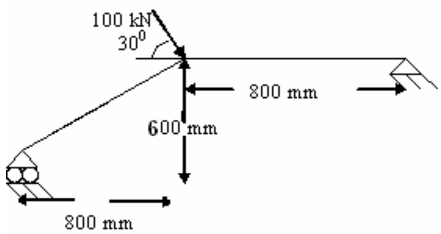
2 An axial load P =300X10^{3}N is applied at 20^0 C to the rod as shown in Figure below. The temperature is the raised to 60^0 C a) Assemble the K and F matrices b) Determine the nodal displacements and stresses



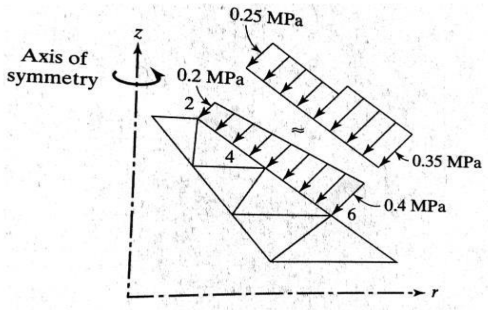
3 Determine the stiffness matrix, stresses and reactions in the truss structure shown in Figure.



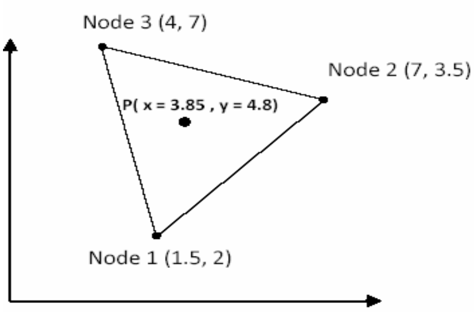
4 Estimate the displacement vector, strains, stresses and reactions in the truss structure shown below in figure. Take A = 1000 mm^2 and E = 200 GPa



5 An axisymmetric body with a linearly distributed load on the conical surface is shown in Fig. Determine the equivalent point loads at nodes 2, 4 and 6.



6 An Isoparametric constant strain triangular element is shown in Figure.
i) Evaluate the shape functions N1, N2 and N3 at an intermediate point P for the triangular element.
ii) Determine the Jacobean of transformation J for the element.



7 Describe heat transfer analysis for straight fin

8 Obtain the Eigen values and Eigen vectors for the cantilever beam of length 2 m using consistent mass for translation DOF with E = 200GPa, ρ =7500kg/m^3.

******

Code No: **R17A0320** **R17**

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Regular Examinations, October/November 2020

Finite Element Methods

(ME)

Roll No										

Time: 2 hours **Max. Marks: 70**

Answer Any **Four** Questions

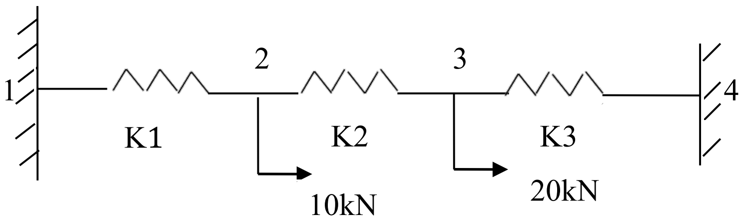
All Questions carries equal marks.

****

1 a Discuss how finite element method is evolved in the engineering field.

b Discuss the advantages and disadvantages of Finite Element Method

2 Solve for the nodal displacement and support reactions, using the principle of Min. Potential Energy approach for the system shown in Figure.



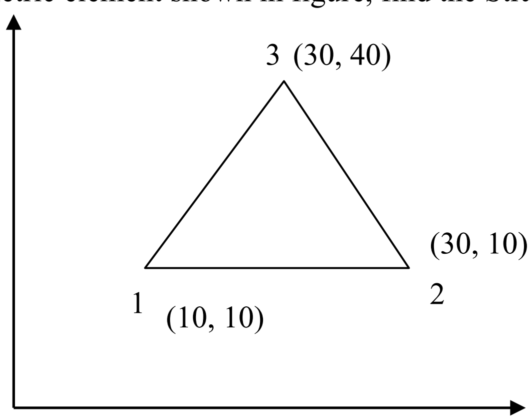
K1=1200kN/m
K2=1800kN/m
K3=1500kN/m

3 Derive stiffness matrix for a Truss bar Element

4 Derive the stiffness matrix for a Three noded CST Element.

5 a What is an axi-symmetric problem?

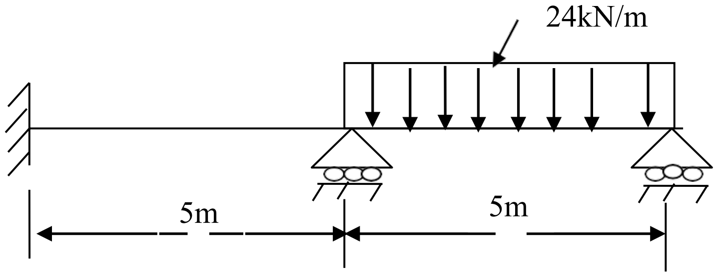
b For the Axi-symmetric element shown in figure, find the Strain-Displacement Matrix.



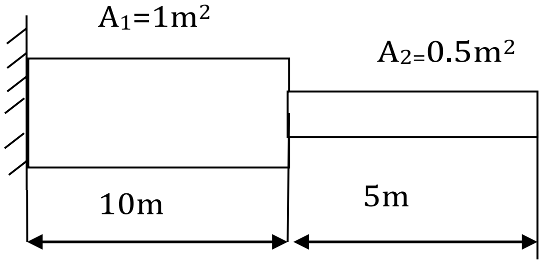
6 Use Gaussian Quadrature to obtain the exact value of the integral

$$f(x) = \int_{-1}^{1} \frac{1}{1+x^2} + 2x - \sin x$$

7 For the beam loaded as shown in figure, determine the slope at the simple supports. Take E=200GPa, I=$4 \times 10^6 m^4$.



8 Determine the Eigen values and Eigen vectors for the beam shown in figure



E=$30 \times 10^5 N/m^2$

P=$0.283 kg/m^3$

****

R17

Code No: R17A0320

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Supplementary Examinations, February 2021

Finite Element Method

(ME)

Roll No										

Time: 2 hours 30 min **Max. Marks: 70**

Answer Any **Five** Questions

All Questions carries equal marks.

***

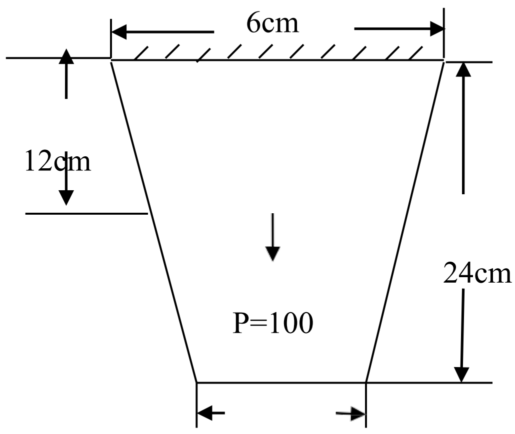
1 Derive the equations of equilibrium of a 3-Dimensional stressed body. **[14M]**

2 Consider the thin (steel) plate shown in figure. The plate has a uniform thickness t=10mm, Young's modulus $E=20x10^9 N/m^2$. **[14M]**

a) Using the elimination approach, solve for the global displacement vector

b) Evaluate the stresses in each element.

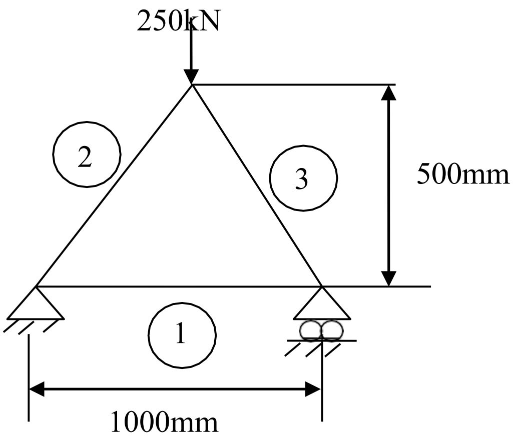
c) Determine the reaction force at the support.



3 Consider a three bar truss as shown in figure. It is given that $E=2X10^5 N/mm^2$. Calculate the following: **[14M]**

(i) Nodal displacements (ii)Stress in each member

(iii) Reactions at the support.



Take:
A_1=2000 mm^2
A_2= 2500 mm^2
A_3= 2500 mm^2

4 a What are the elements commonly used in the analysis of 2-Dimensional roblem? **[04M]**

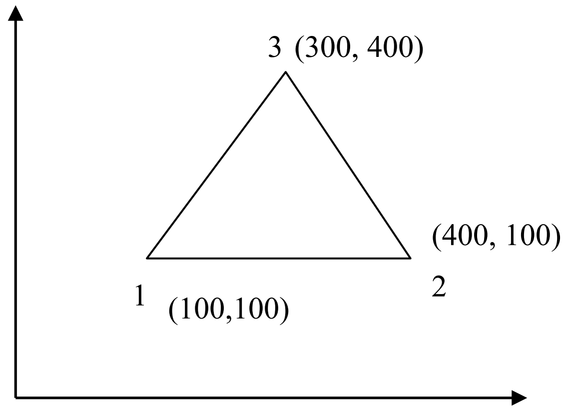
b Derive Strain-Displacement matrix for the 3-noded triangular element. **[10M]**

5 For the plane stress element shown in figure the nodal displacements are **[14M]**

$U_1 = 2$ mm $\quad V_1 = 1$ mm

$U_2 = 1$mm $\quad V_2 = 1.5$mm

$U_3 = 2.5$mm $\quad V_3 = 0.5$mm



Determine the element stresses. Assume $E = 200 GN/m^2$, $\upsilon = 0.3$, t= 10mm.

6 Use Gaussian Quadrature to obtain the exact value for the following integral. **[14M]**

$$\int_{-1}^{1} (r^3 - 1)(s^2 + s) dr.ds$$

7 A wall consists of 4cm thick wood, 10cm thick glass fiber insulation and 1cm thick plaster. If the temperature on the wood and plaster faces are 20°C and -20°C respectively. Determine the temperature distribution in the wall with 1D linear element approach. Assume thermal conductivity of wood, glass and plaster as 0.17, 0.035 and 0.5 W/m-°C. The convective heat transfer coefficient on the colder side of the wall as $25 W/m^2$-°C. **[14M]**

8 Write short note on

(a) Formulation of Finite Element model in dynamic analysis **[4M]**

(b) Eigen vectors for a stepped bar. **[10M]**

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R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Supplementary Examinations, February 2021

Finite Element Method

(ME)

Roll No										

Time: 2 hours 30 min **Max. Marks: 75**

Answer Any **Five** Questions

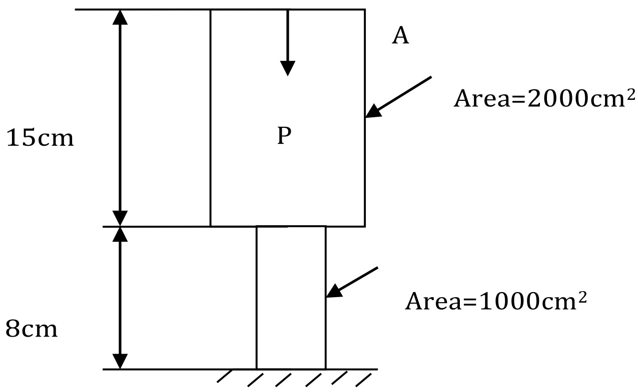
All Questions carries equal marks.

***

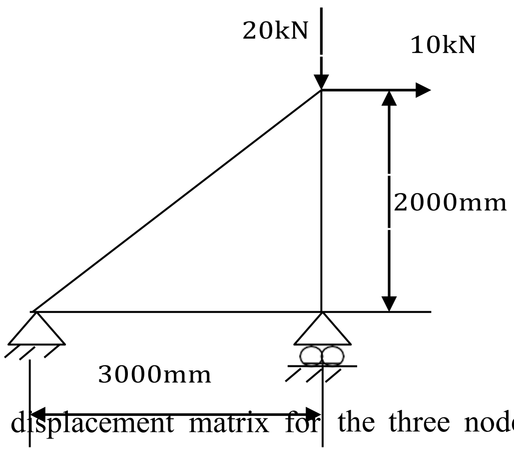
1 a Enumerate thegeneralized procedure involved in Finite Element Method **[10M]**

b Discuss the different engineering applications of Finite Element Method **[05M]**

2 For the vertical bar shown in figure, find the deflection at 'A' and the stress distribution. Use E=150MPa and P=100 KN . **[15M]**



3 Consider the plane truss shown in figure, determine the nodal displacements, Element forces and support reactions. Take E=2X10^5 N/mm^2; A= 1500mm^2. **[15M]**



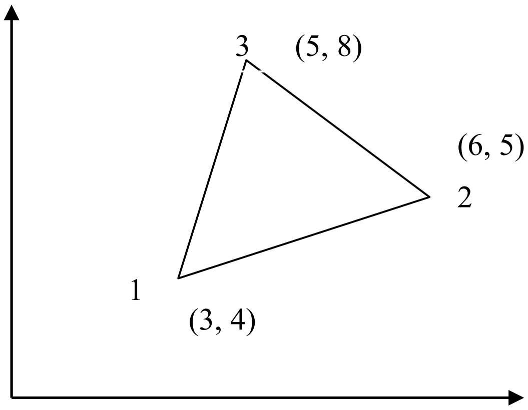
4 Compute Nodal displacement matrix for the three noded triangular element **[15M]**

shown in figure and also determine the element strains, if the nodal displacements are given as

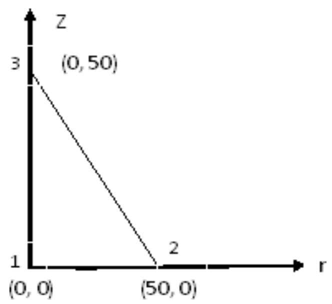
U_1= 0.002 cm V_1= 0.001cm E=200 Gpa & υ = 0.25

U_2= 0.001 cm V_2= -0.004 cm

U_3 = -0.003 cm V_3= 0.007 cm



5 For axi-symmetric element shown in figure, determine the stiffness matrix. Let $E = 2.1x10^5 N/mm^2$ and $\upsilon = 0.25$. The co-ordinates shown in figure are in millimeters. **[15M]**



6 Derive the stiffness matrix for a four noded isoparametric quadrilateral element. **[15M]**

7 Estimate the temperature distribution in a fin whose cross section is 10mmx10mm and 500mm long. Take thermal conductivity as 50W/m-k and convective heat transfer coefficient as $75W/m^2k$ at 25°C. The base temperature is assumed to be constant and its value may be taken as 900°C. And also calculate heat transfer rate? **[15M]**

8 a Distinguish between lumped mass and consistent mass matrices **[06M]**

b Derive the consistent mass matrix for an one dimensional bar element. **[09M]**

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R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Regular/supplementary Examinations, April/May 2019

Finite Element Methods

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B
Part A is compulsory which carriers 25 marks and Answer all questions.
Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

***

PART-A (25 Marks)

1). a What is meant by finite Element method **[2M]**
b Name the weighted residual techniques? **[3M]**
c Write down the expression of stiffness matrix for a truss element. **[2M]**
d Define plane strain problem. **[3M]**
e What is CST element? **[2M]**
f Write down the shape functions for an axisymmetric triangular element. **[3M]**
g Write the governing equation for a steady flow heat conduction. **[2M]**
h Write down the expression of stiffness matrix for a beam element. **[3M]**
i What is meant by discretization and assembling? **[2M]**
j What is the difference between static and dynamic analysis? **[3M]**

PART-B (50 MARKS)

SECTION-I

2 Describe advantages, disadvantages and applications of finite element analysis. **[10M]**

OR

3 The following equation is available for a physical phenomena
$\frac{d^2y}{dx^2}$ -10x^2=5; 0<x<1, Boundary Conditions; y(0) =0, y(1) =0, Using Galarkin method of weighted residual find an approximate solution of the above differential equation. **[10M]**

SECTION-II

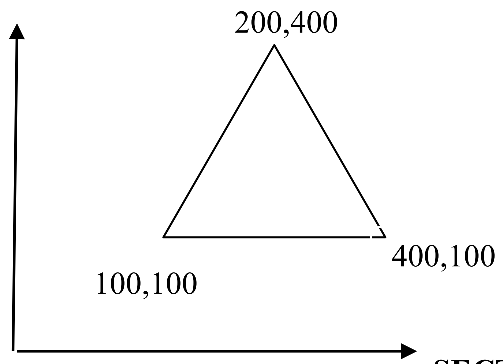
4 For the two bar truss shown in figure, determine the displacement at node 1 and stresses in element2, Take E=70GPa, A= 200mm^2. **[10M]**

OR

5 For the plane stress element shown in figure the nodal displacements are
U_1= 2.0mm, V_1=1.0mm

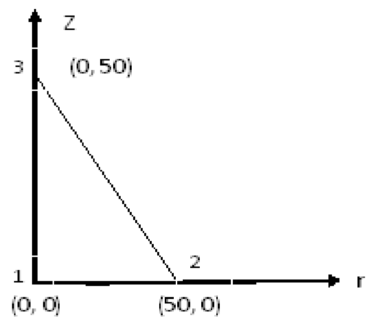
U_2 = 1.

0 mm, V_2= 1.5mm, U_3= 2.5mm, V_3=0.5mm, Take E= 210GPa, ν= 0.25, t=10mm. Determine the strain-Displacement matrix [B]. **[10M]**



SECTION-III

6 For axisymmetric element shown in figure, determine the strain-displacement matrix. Let E = $2.1x10^5$N/mm^2 and ν= 0.25. The co-ordinates shown in figure are in millimeters. **[10M]**



OR

7 Evaluate the following integral using Gaussian quadrature, so that the result is exact. **[10M]**

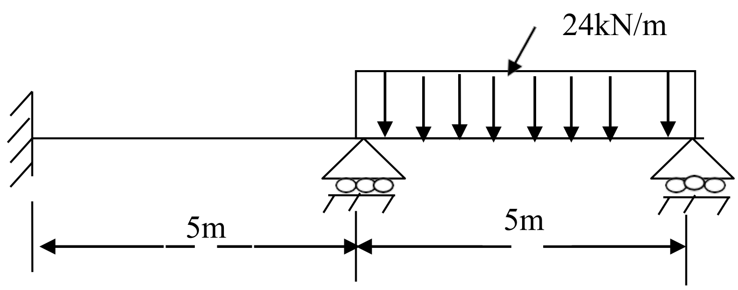
$f(r) = \int_{-1}^{1}(\frac{1}{1+x^2}$ +2x –sinx) dx

SECTION-IV

8 Estimate the temperature distribution in a fin whose cross section is 15mm X 15mm and 500mm long. Take Thermal conductivity as 50W/m-k and convective heat transfer coefficient as 75 W/m^2-k at 25°C. The base temperature is assumedto be constant and its value may be taken as 900°C. And also calculate the heat transfer rate? **[10M]**

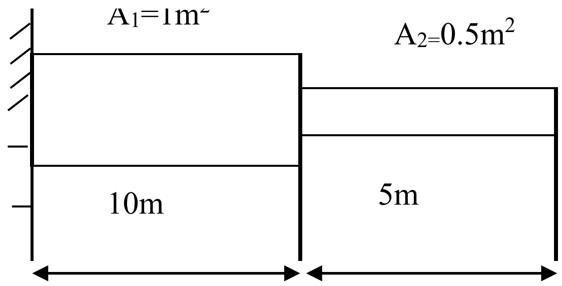
OR

9 For the beam loaded as shown in figure, determine the slope at the simple supports. Take E=200GPa, I=$4x10^6 m^4$. **[10M]**



SECTION-V

10 Determine the Eigen values and Eigen vectors for the beam shown in figure



$E=30x10^5 N/m^2$
$\rho=0.283kg/m^3$

[10M]

OR

11 Write short note on **[10M]**

(a) Eigen vectors for a stepped beam
(b) Evaluation of Eigen values.

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R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Regular Examinations, April/May 2018

Finite Element Method

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B
Part A is compulsory which carriers 25 marks and Answer all questions.
Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

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PART- A

1a. What is the shape function? Give its practical importance. [2]

b) Briefly discuss the Gherkin's approach in solving FEA problems [3]

c) Define is axisymmetric element with 2 practical applications [2]

d. What are the differences between plane stress and plane strain problems [3]

e. Briefly discuss the advantages of Axisymmetric Elements [2]

f. Describe the shape functions in natural coordinates for 2-D Quadrilateral element. [3]

g. Write the governing equation for a steady flow heat conduction [2]

h. Write short notes on applications of FEM [3]

i. What are the practical importance of Eigen values and Eigen vectors [2]

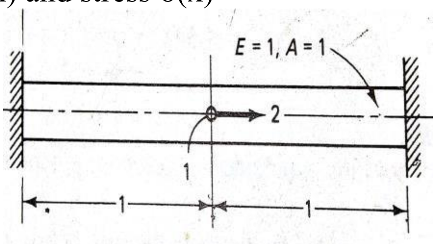
j. Write the Gradient matrix[B] for CST element. [3]

PART – B

10 * 5 = 50 Marks

2. SECTION-1 [5]

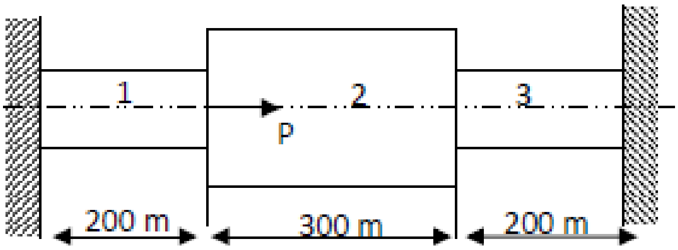
(a) A rod fixed at its ends is subjected to a varying body force as shown in Figure.1. Use the Rayleigh-ritz method with an assumed displacement field $u=a_0+a_1x+a_2x^2$ to determine displacement u(x) and stress σ(x)



(b) Write the Potential function for a continuum under all possible loads and indicate all the variables involved. Also express the total potential of general finite element in terms of nodal displacements [5]

OR

3. An axial load P = 200X10^3 N is applied on a bar shown in figure, determine nodal displacements, stress in each material and reaction forces. If A_1= 2400 mm^2, A_2= 600 mm^2, A_3= 2000 mm^2,E_1=70GPa,E_2=200GPa, E_3=67GPa [10]



4. [5]

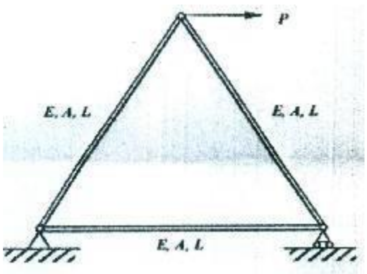
SECTION - II

(a) Derive the B Matrix (relating strains and nodal displacements) for an iso parametric triangular element with linear interpolation for the geometry as well as field variables.

b) Explain why the above element is popularly known as CST. Discuss about the advantages and disadvantages of the element [5]

OR

5. For the truss shown in figure establish the element stiffness matrices and assemble the global stiffness matrix for the active degrees of freedom and determine a) Nodal displacements b) Stress in the members and c) The reaction at the roller support, Take E= 100 Gpa. Area of c/section = 100 mm^2 Length = 100 cm, P = 100 kN. [10]

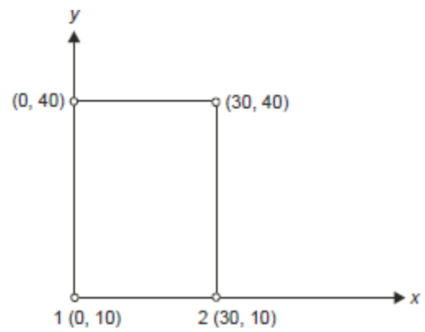


SECTION-III

6. Derive the B Matrix (relating strains and nodal displacements) for an axi-Symmetric iso parametric triangular element with linear interpolation for the geometry as well as field variables. [10]

OR

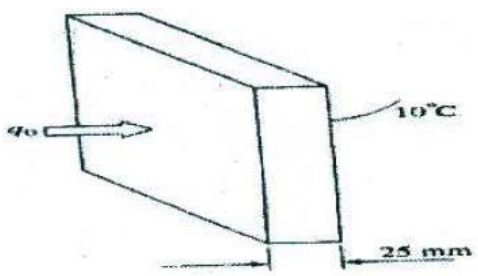
7.(a) Consider a quadrilateral element as shown in figure, Evaluate Jacobian matrix and strain-Displacement matrix at local coordinates ξ =0.5, η = 0.5. [7]



(b) Evaluate the integral $\int_{-1}^{+1} [3e^x + 2x^2 + \frac{1}{(3x+4)}]\, dx$
using one point and two point Gauss quadrature. [3M]

SECTION-IV

8. Heat is entering into a large plate at the rate of q_0=-300w/m2 as shown in Figure, the plate is 25 mm think. The outside surface of the plate is maintained at a temperature of10 ^{0}C. Using two finite elements, solve for the vector of nodal temperatures T, thermal conductivity k=1.0 w/m^{0}c [10]

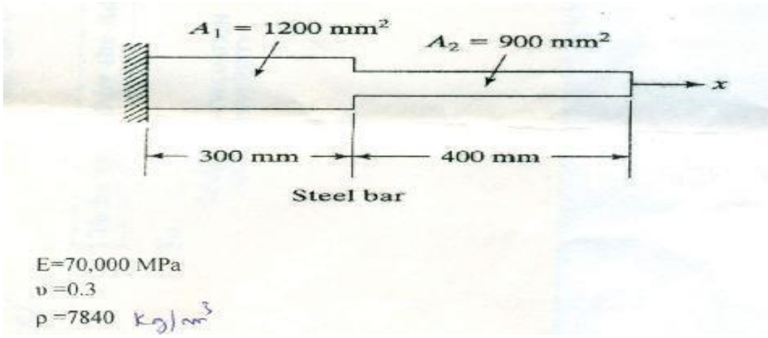


OR

9. Estimate the temperature profile in **a** fin of diameter 25 mm, whose length is 400mm. The thermal conductivity of the fin material is 50 W/m K and heat transfer coefficient over the surface of the fin is 50 W/m^2 K at 30^{0}C. The tip is insulated and the base is exposed to a temperature of 150 ^{0}C. Evaluate the temperatures at points separated by 100 mm each. [10]

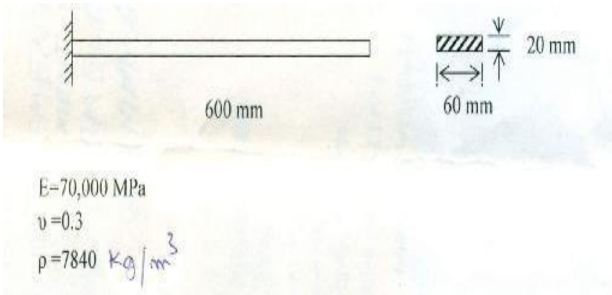
SECTION-V

10. Consider axial vibration of the steel bar shown in Fig. a) Develop the global stiffness and mass matrices b) By hand calculations, determine the lowest natural frequency and mode shape 1and 2 [10]



OR

11. Write the step by step procedure to determine the frequencies and nodal displacements of the steel cantilever beam shown in Fig. [10]



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R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester supplementary Examinations, Nov/Dec 2018

Finite Element Methods

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B
Part A is compulsory which carriers 25 marks and Answer all questions.
Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

PART – A

1.a. Briefly discuss weighted residual method for giving approximate solutions for complicated domains [2M]
b. Write the stiffness matrix for 1-d element with linear interpolation functions [3M]
c. Differentiate iso-parametric, sub-parametric, and super parametric elements? [2M]
d. What is the difference between plane truss and space truss? [3M]
e. What are the uses of natural coordinates in 2d- Quadrilateral elements [2M]
f. What are the suitable applications of axi-symmetric elements in FEM? [3M]
g. Write the governing equation for FEA formulation for a fin [2M]
h Express the stiffness matrix for a 1-D conduction problem [3M]
i. What do you understand by mode shapes? [2M]
j. How principle of minimum potential energy is useful in dynamic analysis of systems [3M]

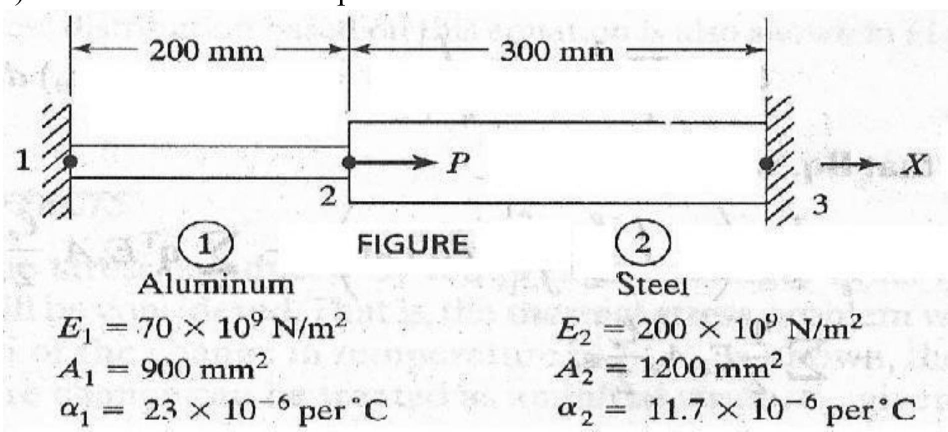
PART – B 10 * 5 = 50 Marks

SECTION-I

2. Derive the equations equilibriums for 3-D body [10M]

OR

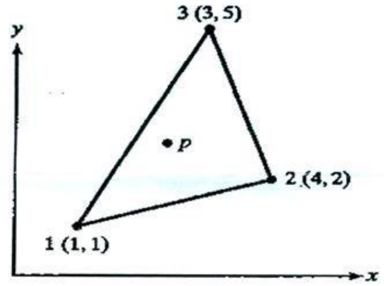
3. An axial load P=300X103N is applied at 200 C to the rod as shown in Figure below. The temperature is the raised to 600 C . [10M]
a) Assemble the K and F matrices.
b) Determine the nodal displacements and stresses.



SECTION-II

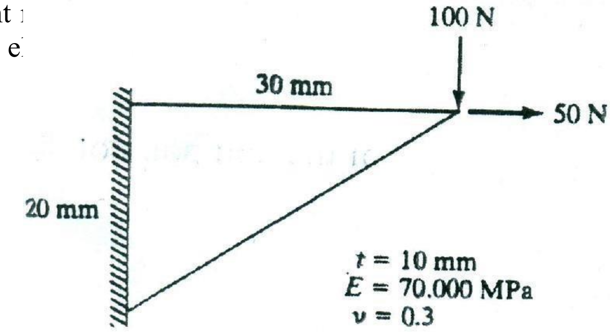
4. a) Write the difference between CST and LST elements [3M]

b) For point P located inside the triangle shown in the figure below the shape functions N1 and N2 are 0.15 and 0.25, respectively. Determine the x and y coordinates of point P. [7M]



OR

5 For the configuration shown in Fig. determine the deflection at the point of load application using a one-element : used, comment on the stress values in the e [10M]



S

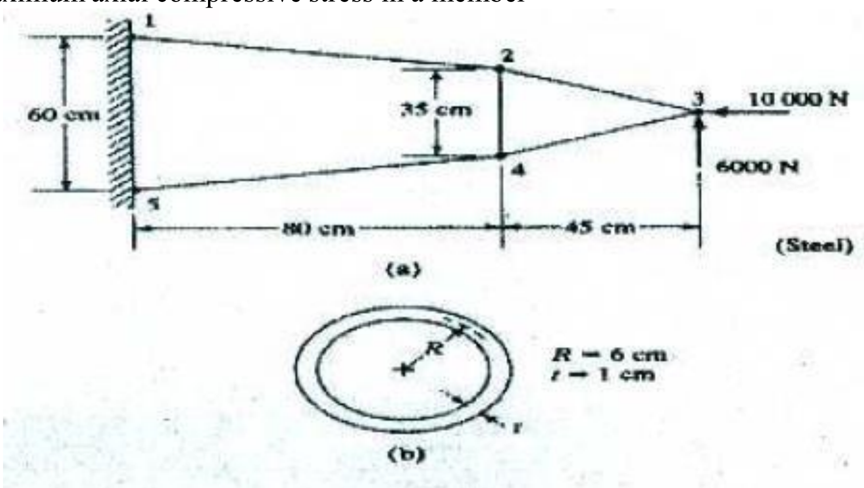
6. Derive the strain displacement matrix for axisymmetric triangular element Discuss advantages of axisymmetric modelling in FEM [10M]

OR

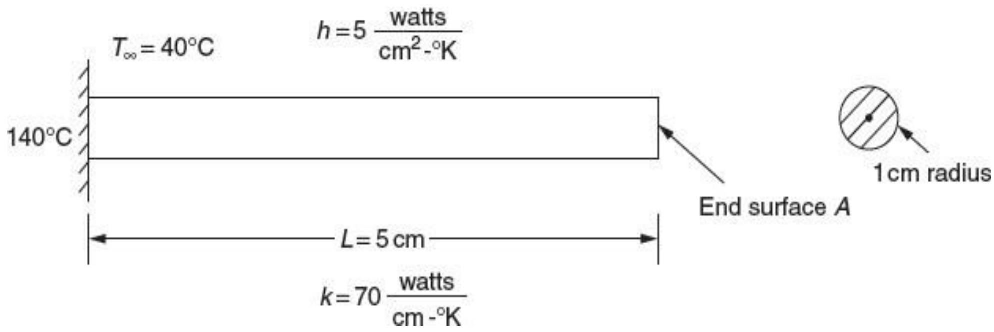
7. Figure shows a five – member steel frame subjected to loads at the free end. The cross section of each member is a tube of wall thickness t=1 cm and mean radius=6cm. Determine the following: [10M]

a) The displacement of node 3 and

b) The maximum axial compressive stress in a member



8. Find the temperature distribution in the one-dimensional fin shown in Figure below using two finite elements. [10M]



OR

9. (a) A 20-cm thick wall of an industrial furnace is constructed using fireclay bricks that have a thermal conductivity of k = 2 W/m-°C. During steady state operation, the furnace wall has a temperature of 800°C on the inside and 300°C on the outside. If oneof the walls of the furnace has a surface area of 2 m^2 (with 20-cm thickness), find the rate of heat transfer and rate of heat loss through the wall. [5M] [5M]
(b) A metal pipe of 10-cm outer diameter carrying steam passes through a room. The walls and the air in the room are at a temperature of 20°C while the outer surface of the pipe is at a temperature of 250°C. If the heat transfer coefficient for free convection from the pipe to the air is h = 20 W/m^2-°C find the rate of heat loss from the pipe.

10. For the two-bar truss shown in Figure below, determine the nodal displacements, element stresses and support reactions. A force of P=1000kN is applied at node-1. Assume E=210GPa and A=600mm2 for each element. [10M]

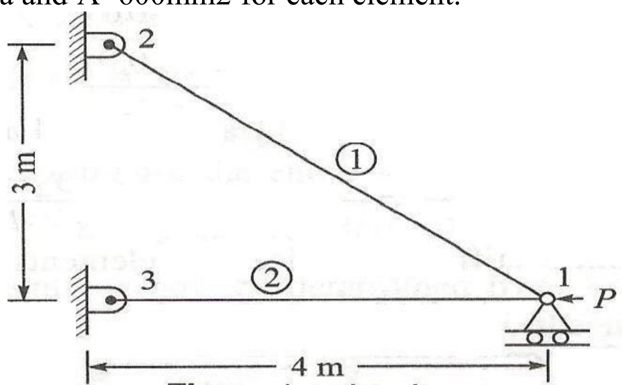


Figure two-bar truss.

OR

11. A bar of length 1 m; cross sectional area 100 mm2; density of 7 gm/cc and Young's modulus 200Gpa is fixed at both the ends. Consider the bar as three bar elements and determine the first two natural frequencies and the corresponding mode shapes. Discuss on the accuracy of the obtained solution [10M]

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R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Supplementary Examinations, December 2019

Finite Element Method

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B
Part A is compulsory which carriers 25 marks and Answer all questions.
Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

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PART-A (25 Marks)

1). a	What is meant by Engineering analysis and specify its Types	**[2M]**
b	Explain finite element method?	**[3M]**
c	Draw a plane truss structure.	**[2M]**
d	What are the characteristics of a truss?	**[3M]**
e	Define shape function.	**[2M]**
f	List any four two dimensional elements.	**[3M]**
g	What is Fourier's law?	**[2M]**
h	Discuss the types of heat transfer	**[3M]**
i	What is consistent mass matrix?	**[2M]**
j	Define Eigen values?	**[3M]**

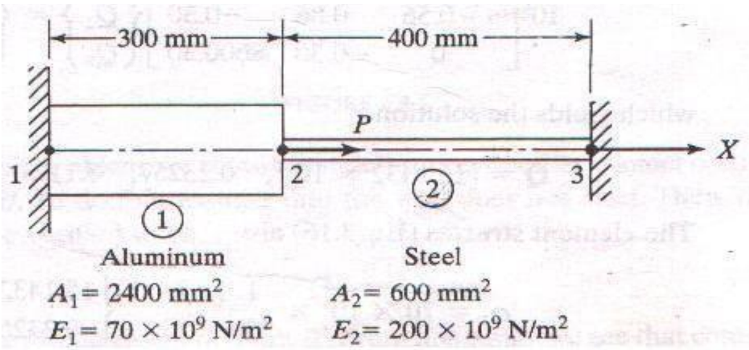
PART-B (50 MARKS)

SECTION-I

2 Explain the concept of FEM briefly and outline the steps involved in FEM along with remembers. **[10M]**

OR

3 Consider the following fig. An axial load P=200 KN is applied as shown. Using penalty approach for handling boundary conditions, do the following **[10M]**
a) Determine the nodal displacements.
b) Determine the stress in each material.
c) Determine the reaction forces.


300 mm
400 mm
P
X
1
2
3
1
2
Aluminum
Steel
A1 = 2400 mm2
E1 = 70 × 10^9 N/m2
A2 = 600 mm2
E2 = 200 × 10^9 N/m2

SECTION-II

4 For the two-bar truss shown in Figure below, determine the nodal displacements, element stresses and support reactions. A force of P=1000kN is applied at node-1. Assume E=210GPa and A=600mm^2 for each element. **[10M]**

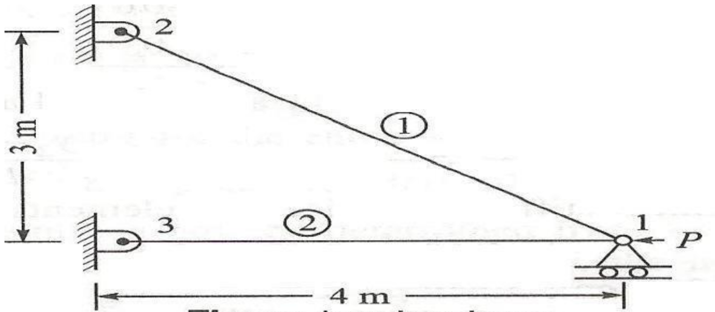


Figure two-bar truss.

OR

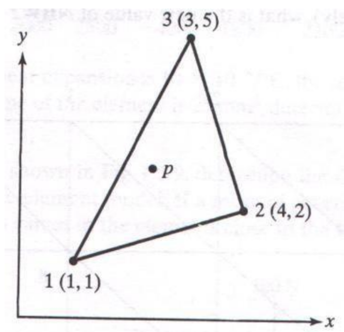
5 a).Explain Iso-parametric, sub-parametricand super- parametric elements **[6M]**
b) Advantages of iso-parametric elements **[4M]**

SECTION-III

6 Explain the concept of numerical integration and its utility in generating Isoperimetric finite element matrices. **[10M]**

OR

7 For the point P located inside the triangle, the shape functions N_1 and N_2 are 0.15 and 0.25, respectively. Determine the x and y coordinates of P. **[10M]**

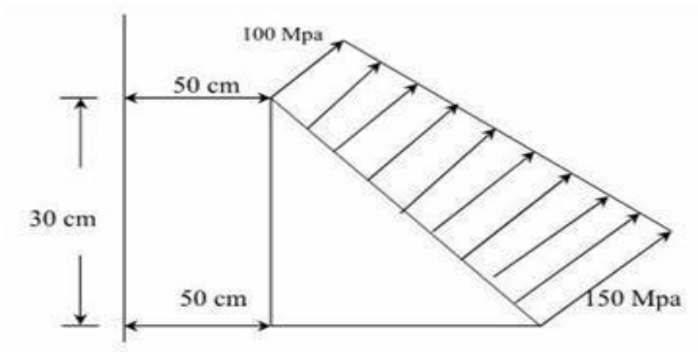


SECTION-IV

8 Estimate the temperature profile in **a** fin of diameter 25 mm, whose length is 500mm. The thermal conductivity of the fin material is 50 W/m K and heat transfer coefficient over the surface of the fin is 40 W/m^2 K at 30^{0}C. The tip is insulated and the base is exposed to a temperature of 150 ^{0}C. Evaluate the temperatures at points separated by 100 mm each. **[10M]**

OR

9 An axi-symmetric triangular element is subjected to the loading as shown in fig. the load is distributed throughout the circumference and normal to the boundary. Derive all the necessary equations and derive the nodal point loads. **[10M]**



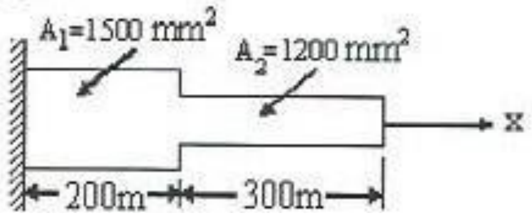
SECTION-V

10 Explain the following with examples: **[10M]**

a) Lumped mass matrix. b) Types of vibrations.

OR

11 Determine the natural frequencies and mode shapes of a stepped bar shown in figure below. Assume E=300GPa and density is 7800 Kg/m^3. **[10M]**



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